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## STUDIES ON THE THEORY OF STRUCTURES (COLLECTION OF ARTICLES)



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## ABSTRACT

THE BOOK IS COMPRISED OF

(U) A series of articles devoted to problems of structural dynamics, the determination of critical velocities of moving loads, the vibrations of multistory frame buildings and the dynamic design of arches. Discussions are presented on the design of various structures for stability in the presence of linear and nonlinear couplings, within and beyond the elastic limit. Quantitative estimates of critical forces are provided.

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Candidate of Technical Sciences,  
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*A number of articles in this collection deal with problems of dynamics of structures, determining critical speeds of moving loads, vibrations of frame-type multistory buildings and dynamic design of arches.*

*These articles illuminate the problems of designing various structures for stability in the presence of linear and nonlinear couplings, when the structures function within and outside the elastic limit, for different load behaviors in the process of structure deformation; certain quantitative estimates of critical forces are given.*

*The articles in this collection have as their aim aiding in practical adaption into the design of structures of more refined methods, refinement and perfection of existing methods, and providing answers to certain questions which arise in calculating and designing structures.*

*This book is intended for design engineers of industrial, civil and other structures, instructors, degree candidates and students of technical construction colleges, as well as for staff members of construction scientific research institutes.*

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# STABILITY OF ECCENTRICALLY COMPRESSED BARS WITH AN ASYMMETRICAL H AND T CROSS SECTION IN THE ELASTOPLASTIC RANGE

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## §1. Assumptions Used and the Statement of the Problem

Stability in the plane of forces of eccentrically compressed bars with asymmetrical H and T sections is studied on the basis of the following assumptions: 1) the material is an ideal elastoplastic with a yield plateau of unlimited length; 2) the bars are hinged at the ends and loaded by a longitudinal compressive force  $N$ , applied in the plane of the web with an initial eccentricity  $a$  (Fig. 1); 3) the load is applied once, and elastoplastic deformations function from the instant of application of the compressive force to the ensuance of the critical state; 4) the bars are fastened so that the deflection remains plane; 5) the flange thickness of the H and T sections is assumed to be negligible in comparison with the height of the cross section; 6) use is made of an approximate expression for the bar's curvature on deflection, and the shortening of the bar due to axial compression is not taken into account; 7) it is assumed that the bent axis of the bar at the instant of buckling is in the shape of a sinusoidal half-wave

$$y = y_m \sin \frac{\pi x}{l}. \quad (1)$$

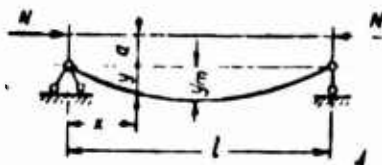


Fig. 1

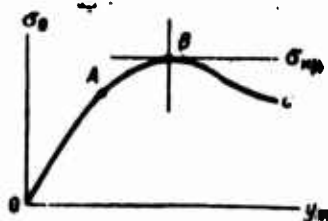


Fig. 2

Here  $x, y$  are the coordinates of a point on the axis of the deformed bar,  $y_m$  is the maximum midspan deflection and  $l$  is the length of the bar.

The behavior of an eccentrically compressed bar from an elastoplastic material is illustrated by the graph (Fig. 2). The maximum midspan deflection  $y_m$  is laid off on the ordinate axis, and the axial stress  $\sigma_0 = N/F$  is laid off on the abscissa axis ( $F$  is the cross-sectional area of the bar). Point A of the graph corresponds to the dangerous state, i.e., to the boundary between elastic and elastoplastic stages in the material's functioning. At point B, the bar buckles, axial stress  $\sigma_0$  reaches its maximum, which is called the critical value  $\sigma_{kr}$ . The analytic condition for buckling is expressed by the equation

$$\frac{d\sigma_0}{dy_m} = 0. \quad (2)$$

If we place some stress diagram in the middle section of the bar (consequently, also an axial stress  $\sigma_0$ , corresponding to this diagram), then the equilibrium conditions will yield a relationship such as  $l = l(y_m)$ . From the ensemble of bars with different  $l$  and  $y_m$  which satisfy this equilibrium condition, the longest bar will be in the critical state. From this we get the criterion for buckling

$$\frac{dl}{dy_m} = 0 \quad \text{or} \quad \frac{dl}{d\mu} = 0, \quad (3)$$

which is also used in the present study. Here  $\mu$  is any parameter which varies monotonically with changes in  $y_m$ .

Let  $F_1$  be the area of the H-beam flange put under compression by the bending, let  $F_2$  be the area of the opposite flange and  $F_{st}$  the web area. We introduce two parameters of shape of cross section of an asymmetrical H beam, the section area parameter

$$\alpha = F_{st}/F_1 \quad (4)$$

and the asymmetry parameter

$$\beta = F_2/F_1. \quad (5)$$

When H sections bend in the direction of the flange, parameter  $\alpha$  is understood to denote the ratio of the web area to the area of the flange under tension due to flexure. We also introduce the parameter

$$\gamma = \alpha + 3 \frac{\alpha + \alpha\beta + 4\beta}{\alpha + \beta + 1}. \quad (6)$$

The stability of eccentrically compressed elastoplastic bars with different cross-sectional shapes was studied, using similar starting assumptions, by K. Jezek [1], V.V. Pinadzhyan [2], A.V. Gemmerling [3] and other authors. In this work, the use of the assumption on small flange thickness and the utilization of the shape parameters  $\alpha$  and  $\beta$  made it possible to obtain appreciably simpler and compact analytical relationships. The results of stud-

ies allow estimating the effect of material distribution over the cross section on the bar's stability.

An analysis of the error introduced by the assumption of small flange thickness is given in Reference [4] for a symmetrical H section.

## 2. Stability of Bars with an Asymmetrical H Section

**Critical state 1.** Depending on the shape of the stress diagram in the H-profile midsection on buckling (of the second kind) it is possible to have seven different critical states. We start our study with critical state 1, for which yielding in the midspan bar cross section extends to a part of the thickness of the compressed flange (Fig. 3). Since it was assumed that the flange thickness is negligible as compared with the section's height, then in this case the critical section will become identical with the dangerous cross point, for which the stress in the fibers reaches the yield point  $\sigma_t$ :

$$\sigma = \sigma_0 \left( 1 + \frac{MF}{NW_1} \right) = \sigma_t, \quad (7)$$

where  $W_1$  is the section modulus of the shape.

The bending moment at midspan is  $M = N(a + y_m)$ . We relate  $y_m$ , the maximum midspan deflection to the bar's flexibility  $\lambda = l \sqrt{F/I}$ . For this purpose, we shall substitute into the differential equation of elastic flexure

$$y'' = -\frac{M}{EI} = -\frac{N}{EI} (a + y) \quad (8)$$

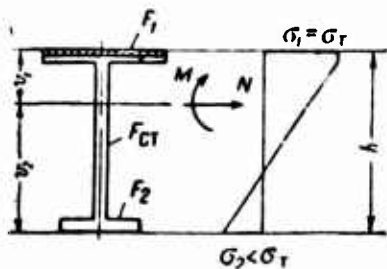


Fig. 3

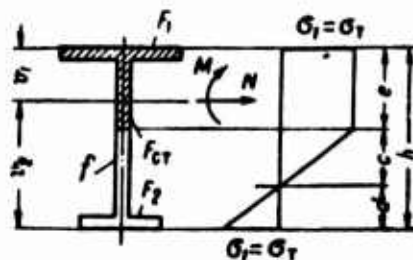


Fig. 4

the previously assumed sinusoidal expression (1) for the deflection curve and we shall then set  $x = l/2$ ,  $y = y_m$ . We denote by  $\lambda_e$  that bar flexibility for which the Eulerian critical stress is equal to axial stress  $\sigma_0$  under consideration, whence

$$\lambda_e = \pi \sqrt{EI/\sigma_0}, \quad (9)$$

and the maximum deflection is

$$y_m = \frac{a}{\lambda_0^2/\lambda^2 - 1} \quad (10)$$

On the basis of previously derived relationships, the flexibility of a bar in the critical state is

$$\lambda = \sqrt{1 - \nu \frac{\sigma_r}{\sigma_r - \sigma_{kp}}} \lambda_0 \quad (11)$$

where  $\nu$ , the relative eccentricity, is the ratio of the force arm to the core distance

$$\nu = \frac{a}{k} = \frac{aF}{W_1} = 6 \frac{a + 2\beta}{\gamma} \cdot \frac{a}{h} \quad (12)$$

Formula (11) establishes, for a given eccentricity, a relationship between  $\lambda$ , the bar's flexibility, and the axial stress  $\sigma_0 = \sigma_{kp}$ , which has corresponding to it the stress diagram shown in Fig. 3.

**Critical state 2.** The yielding zone extends to the compressed flange and to that part of the web adjoining it (Fig. 4). We denote by  $f$  that part of the web area which is within the limits of the elastic core; then

$$\mu = f/F_1 \quad (13)$$

the parameter of the stressed state, will characterize the degree of development of plastic deformations. When  $\mu = \alpha$  the section's operation is entirely elastic, while when  $\mu = 0$  its operation is entirely plastic.

The conditions of equilibrium between the external and internal forces in the bar's midspan section can be used to determine distance  $c$  from the zero point of the stress diagram to the boundary between the elastic core and the yielding zone

$$\frac{c}{h} = \frac{\sigma_r}{\sigma_r - \sigma_0} \cdot \frac{\mu(\mu + 2\beta)}{2\alpha(\alpha + \beta + 1)} \quad (14)$$

and the maximum deflection

$$\frac{y_m}{h} = \frac{\sigma_r - \sigma_0}{2\sigma_0} \left[ \frac{\alpha + 2}{\alpha + \beta + 1} - \frac{2\mu^2}{3\alpha(\mu + 2\beta)} \right] - \frac{a}{h} \quad (15)$$

The differential equation of elastoplastic deflection has the form

$$y'' = - \frac{\sigma_r}{Ec} \quad (16)$$

Substituting here Expression (1) and then setting  $x = l/2$ ,  $y = y_m$ , we find

$$l^2 = \pi^2 \frac{E}{\sigma_r} (y_m c) \quad (17)$$

This equation characterizes the equilibrium states of the bar. For a given axial stress  $\sigma_0$ , the application of Criterion (3) yields the following condition for buckling:

$$\frac{d(y_{mc})}{d\mu} = 0. \quad (18)$$

Making use of Relationships (14) and (15), we represent Condition (18) in the form

$$\frac{d}{d\mu} \left[ \mu(\mu + 2\beta) p_1 - \frac{\mu^2}{3} \right] = 0, \quad (19)$$

where

$$p_1 = \frac{a}{2} \left( \frac{a+2}{a+\beta+1} - \frac{a}{h} \cdot \frac{2\sigma_0}{\sigma_T - \sigma_0} \right). \quad (20)$$

Differentiating we get the quadratic equation

$$\mu^2 - 2p_1\mu - 2p_1\beta = 0,$$

whence we find the critical value of the parameter of interest

$$\mu_{kp} = p_1 + \sqrt{p_1(p_1 + 2\beta)}. \quad (21)$$

On the basis of the above relationship we eliminate parameter  $\mu$  from Relationships (14) and (15) and introduce the notation

$$\Phi(p, \beta) = 4p[p(p + 3\beta) + (p + 2\beta)\sqrt{p(p + 2\beta)}]. \quad (22)$$

Relationship (17) yields the following expression for the flexibility of the bar in the critical state

$$\lambda = \frac{\sqrt{\Phi(p_1, \beta)}}{\pi\sqrt{\gamma}} \lambda_0. \quad (23)$$

Here  $p_1$  is determined from Relationship (20) which, after replacing  $\sigma_0$  by  $\sigma_{kr}$  and changing to relative eccentricity  $v$ , takes on the form

$$p_1 = \frac{a}{2} \left( \frac{a+2}{a+\beta+1} - \frac{v}{3} \cdot \frac{\gamma}{a+2\beta} \cdot \frac{\sigma_{kp}}{\sigma_T - \sigma_{kp}} \right). \quad (24)$$

It is most convenient to calculate the coordinates of points on the critical-stress diagram in the following manner. It is assumed that the shape parameters  $a$  and  $\beta$  and relative eccentricity  $v$  are known. We specify an arbitrary value of  $p_1$  and find the flexibility from Formula (23). On the basis of (24), the corresponding critical stress is

$$\sigma_{kv} = \frac{\sigma_T}{K+1}, \quad (25)$$

where

$$K = \frac{v}{3} \cdot \frac{a\gamma}{a+2\beta} \cdot \frac{a+\beta+1}{a(a+2) - 2p_1(a+\beta+1)}. \quad (26)$$

It is possible to eliminate  $p_1$  from the computational relationships in order to express the flexibility in terms of parameter  $\mu_{kr}$ . It follows from Eq. (21)

$$p_1 = \frac{\mu_{kr}^2}{2(\mu_{kr} + \beta)} \quad (27)$$

and for the flexibility in the critical state we find

$$\lambda = \frac{1}{a\sqrt{\gamma}} \sqrt{\mu_{kr}^3 \frac{\mu_{kr} + 4\beta}{\mu_{kr} + \beta}} \lambda_0. \quad (28)$$

Let us establish the limits of applicability of the above relationships. When  $\mu_{kr} = a$  the section functions elastically and the expression for flexibilities corresponding to the boundary between critical states 1 and 2 has the following form, on the basis of (28)

$$\lambda_{12} = \sqrt{\frac{a(a+4\beta)}{(a+\beta)\gamma}} \lambda_0. \quad (29)$$

Another limiting case will exist when the fiber stress of the flange under tension reaches the yield point:  $\sigma_2 = \sigma_T$ . From the condition that  $c/h = (c+d)/2h = \mu_1/2a$  and having resort to (14), we find the corresponding value

$$\mu_1 = \frac{\sigma_T - \sigma_{kr}}{\sigma_T} (a + \beta + 1) - 2\beta. \quad (30)$$

From this we get an expression for flexibilities corresponding to the boundary between states 2 and 3

$$\lambda_{23} = \frac{1}{a\sqrt{\gamma}} \sqrt{\mu_1^3 \frac{\mu_1 + 4\beta}{\mu_1 + \beta}} \lambda_0. \quad (31)$$

When  $\lambda = 0$ , we have  $p_1 = 0$  and the corresponding critical stress is

$$\sigma_{k,0} = \frac{\sigma_T}{K_0 + 1}, \quad K_0 = \frac{\gamma}{3} \cdot \frac{(a + \beta + 1)\gamma}{(a + 2)(a + 2\beta)}. \quad (32)$$

The value of  $p_1$  highest for the given critical state is found from the condition  $\mu_{kr} = a$ . Substituting this value of  $\mu_{kr}$  into Eq. (27), we get  $p_1 = a^2/2(a + \beta)$ .

**Critical state 3.** The plastic compression zone extends to the compressed flange and to its adjoining part of the web, a part of the thickness of the tension flange is simultaneously in the plastic range (Fig. 5). Since the flange thickness was assumed negligible in comparison with the height of the section, then that axial stress is critical for which fiber yielding in the tension flange ensues.

Resorting to equilibrium condition (17) and the condition that  $\sigma_2 = \sigma_T$ , which in this case serves as the buckling criterion, we get for the bar's critical-state flexibility

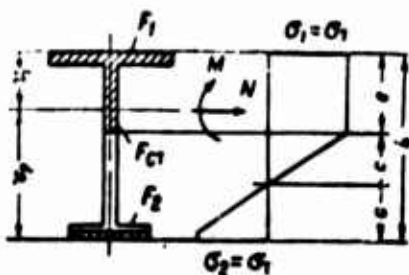


Fig. 5

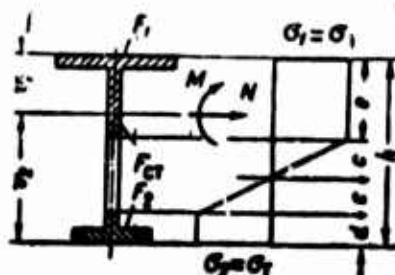


Fig. 6

$$\lambda = \frac{\sqrt{6\mu_1(\mu_1 + 2\beta)\rho_1 - 2\mu_1^3}}{a\sqrt{\gamma}} \lambda_0. \quad (33)$$

Here the value of  $\mu_1$  is determined from Formula (30), from which it follows

$$\sigma_{kp} = \left(1 - \frac{\mu_1 + 2\beta}{a + \beta + 1}\right) \sigma_r. \quad (34)$$

To find points on the critical-stresses graph it is convenient to specify some value of  $\mu_1 < a$ , calculate critical stress  $\sigma_{kr}$  from Formula (34), and then use Formula (24) for  $p_1$  and, finally, determine the bar's flexibility  $\lambda$  from Expression (33).

**Critical state 4.** The plastic compression zone extends to the compressed flange and to a part of the web adjoining it. At the same time, the tension flange and the part of the web adjoining it are in the plastic range (Fig. 6).

The height  $d$  of the plastic tension zone can be found from equilibrium of the forces and stresses in the middle section of the bar

$$\frac{d}{h} = \frac{\sigma_r - \sigma_0}{\sigma_r} \cdot \frac{a + \beta + 1}{2a} - \frac{\beta}{a} - \frac{c}{h}, \quad (35)$$

and also

$$\frac{y_m}{h} = \frac{1}{a + \beta + 1} \left[ \frac{A}{4a} - \frac{a}{3} \left( \frac{c}{h} \right)^3 \right], \quad (36)$$

where

$$A = 2a(a + 2) - (a - \beta + 1)^2 + \left[ 2(1 - \beta^2) - \frac{2}{3} \sqrt{\frac{a(a + \beta + 1)\gamma}{a + 2\beta}} \right] \frac{\sigma_0}{\sigma_r} - (a + \beta + 1)^2 \left( \frac{\sigma_0}{\sigma_r} \right)^2. \quad (37)$$

The buckling state has corresponding to it a maximum of the product  $y_m \sigma$ , which enters Formula (17). In the given case this product is regarded as a function of  $c$ , and the buckling criterion is written in the form

$$\frac{d}{dc} \left\{ \left[ \frac{A}{4a} - \frac{a}{3} \left( \frac{c}{h} \right)^3 \right] \frac{c}{h} \right\} = 0, \quad (38)$$

whence it follows

$$c/h = \sqrt{A}/2a. \quad (39)$$

The critical-state bar flexibility is

$$\lambda = \frac{A^{3/4}}{a\sqrt{\gamma}} \lambda_0. \quad (40)$$

where  $\sigma_0$  in Formula (37) for  $A$  should be replaced by  $\sigma_{kr}$ .

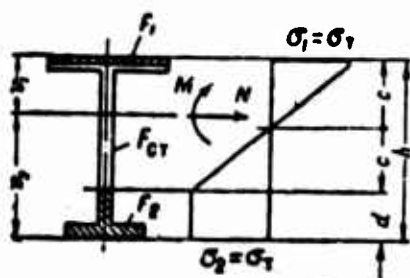


Fig. 7

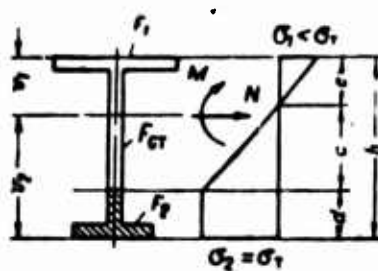


Fig. 8

The boundary between critical states 3 and 4 has corresponding to it  $d = 0$ , and from Expression (35) we find that  $c/h = \mu_1/2a$ , with  $\mu_1$  given by Formula (30). The expression for flexibilities corresponding to the boundary between critical states 3 and 4 takes on the form

$$\lambda_{34} = \frac{\sqrt{\mu_1^3}}{a\sqrt{\gamma}} \lambda_0. \quad (41)$$

The carrying capacity exhaustion due to the fact that the plastic state extends to the entire section takes place when flexibility  $\lambda = 0$ . The corresponding axial stress  $\sigma_{k.0}$  is determined from the condition that  $A = 0$ .

**Critical state 5.** The plastic compression zone extends to a part of the compression flange. The plastic tension zone extends to the tension flange and the part of the web adjoining it (Fig. 7). This case, as well as both following it, is possible for sharply asymmetrical sections (low  $\beta$ ) with substantial eccentricities.

We use the equilibrium conditions. Relationship (35) with the supplementary condition  $2c+d=h$  yields  $c/h = \mu_2/2a$ , with  $\mu_2$  equal to

$$\mu_2 = \frac{\sigma_T + \sigma_{kp}}{\sigma_T} (a + \beta + 1) - 2. \quad (42)$$

On the basis of Relationships (17) and (36), using the notation

$$\rho_1 = \frac{a}{2} \left[ \frac{a + 2\beta}{a + \beta + 1} - \frac{\gamma}{3} \cdot \frac{\gamma}{a + 2\beta} \cdot \frac{\sigma_{kp}}{\sigma_T + \sigma_{kp}} \right] \quad (43)$$

we get for the critical-state flexibility of the bar

$$\lambda = \frac{\sqrt{6\mu_2(\mu_2+2)p_2-2\mu_2^3}}{a\sqrt{\gamma}} \lambda_0 \quad (44)$$

The expression for flexibilities corresponding to the boundary between critical states 4 and 5 is obtained by substituting the value of  $c$  into Relationship (40)

$$\lambda_{45} = \frac{\sqrt{\mu_2^3}}{a\sqrt{\gamma}} \lambda_0 \quad (45)$$

Critical state 6. The yielding extends to the tension flange and to the part of the web adjoining it (Fig. 8).

Conditions of equilibrium of external and internal forces in the middle section of the bar give

$$\frac{c}{h} = \frac{\sigma_T}{\sigma_T + \sigma_0} \cdot \frac{\mu(\mu+2)}{2\alpha(\alpha+\beta+1)} \quad (46)$$

and the maximum midspan deflection

$$\frac{v_m}{h} = \frac{\sigma_T - \sigma_0}{2\sigma_0} \left[ \frac{\alpha + 2\beta}{\alpha + \beta + 1} - \frac{2\mu^2}{3\alpha(\mu+2)} \right] - \frac{a}{h} \quad (47)$$

Buckling criterion (18) results in the equation

$$\frac{d}{d\mu} \left[ \mu(\mu+2)p_2 - \frac{\mu^3}{3} \right] = 0, \quad (48)$$

in which it was denoted

$$p_2 = \frac{\alpha}{2} \left[ \frac{\alpha + 2\beta}{\alpha + \beta + 1} - \frac{a}{h} \cdot \frac{2\sigma_0}{\sigma_T + \sigma_0} \right]. \quad (49)$$

From this we get the critical value of the parameter

$$\mu_{xp} = p_2 + \sqrt{p_2(p_2+2)}. \quad (50)$$

Using the previously introduced notation for a function of two arguments  $\Phi(p, \beta)$ , we will write the expression for the flexibility of a bar in the critical state in the form

$$\lambda = \frac{\sqrt{\Phi(p_2, 1)}}{a\sqrt{\gamma}} \lambda_0 \quad (51)$$

Here, in Expression (49) for  $p_2$ , one should set  $\sigma_0 = \sigma_{kp}$  and then change to relative eccentricity  $v$ , after which we get Formula (43).

To find points on the critical-stresses graph for known values of the shape parameters  $\alpha$  and  $\beta$  and of the relative eccentricity  $v$ , we specify some value of  $p_2$ , calculate the value of the function  $\Phi(p_2, 1)$  on the basis of Relationship (22) and find the bar's flexibility from Formula (51). The corresponding critical

stress is defined by the formula

$$\sigma_{\text{np}} = \frac{\sigma_T}{K-1}, \quad (52)$$

where

$$K = \frac{\nu}{3} \cdot \frac{\alpha\gamma}{\alpha+2\beta} \cdot \frac{\alpha+\beta+1}{\alpha(\alpha+2\beta)-2\nu_1(\alpha+\beta+1)}, \quad (53)$$

The bar's critical-state flexibility can be expressed in terms of  $\mu$ , the stressed-state parameter. Using Relationship (50), we reduce Expression (51) to the form

$$\lambda = \frac{1}{\alpha\sqrt{\gamma}} \sqrt{\mu^3 \frac{\mu+4}{\mu+1}} \lambda_s. \quad (54)$$

The boundaries of applicability of the above relationships are established as follows. The stress in the compression flange reaches the yield point  $\sigma_1 = \sigma_T$  when  $\mu = \mu_2$  from Formula (42). The flexibilities corresponding to the boundary between critical states 5 and 6 are

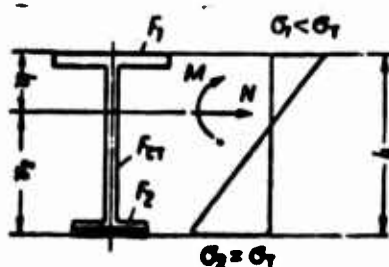


Fig. 9

$$\lambda_{56} = \frac{1}{\alpha\sqrt{\gamma}} \sqrt{\mu_2^3 \frac{\mu_2+4}{\mu_2+1}} \lambda_s. \quad (55)$$

The second limiting case exists when  $\mu = \alpha$ , when the web functions elastically over its entire height. The flexibilities dividing critical states 6 and 7, on the basis of (54), are

$$\lambda_{67} = \sqrt{\frac{\alpha(\alpha+4)}{(\alpha+1)\gamma}} \lambda_s. \quad (56)$$

**Critical state 7.** Yielding extends to part of the thickness of the tension flange (Fig. 9). Since it was assumed that the flange thickness is negligible in comparison with the section height, then the critical state in the given case will be that for which the entire cross section functions elastically, but the fiber stress of the tension flange reaches the yield point

$$\sigma = \sigma_0 \left( \frac{MF}{NW_1} - 1 \right) = \sigma_T. \quad (57)$$

Following the way laid down in deriving Formula (11), we get the following for the critical-state flexibility of the bar

$$\lambda = \sqrt{1 - \nu \frac{\alpha+2}{\alpha+2\beta} \frac{\sigma_{\text{np}}}{\sigma_T + \sigma_{\text{np}}}} \lambda_s. \quad (58)$$

It can be seen from comparison of Formulas (11) and (58) that they yield identical values for the flexibility if  $\sigma_{\text{np}} = \sigma_{0,T}$ , where

$$\sigma_{0,T} = \frac{1-\beta}{\alpha+\beta+1} \sigma_T. \quad (59)$$

This equation, which requires that fiber yielding should set on simultaneously in the compression as well as the tension flange, determines the boundary between critical states 1 and 7.

### §3. Regions on the Critical-Stresses Graphs

We shall depict the results of studies of the stability of an eccentrically compressed H beam with some specified values of shape parameters  $\alpha$  and  $\beta$  on a critical-stresses graph. The bar's flexibility  $\lambda$  is laid off on the abscissa axis, with the critical stress  $\sigma_{kr}$  laid off on the ordinate axis. A family of curves, each of which corresponds to a given specified value of relative eccentricity  $v$ , is plotted on the graph.

The plane of the critical-stresses graph is represented schematically in Fig. 10. This plane is bounded from the top by line  $RT$ , which is the critical-stresses curve for a centrally compressed bar ( $v=0$ ). Rectilinear segment  $RT$  corresponds to a central-

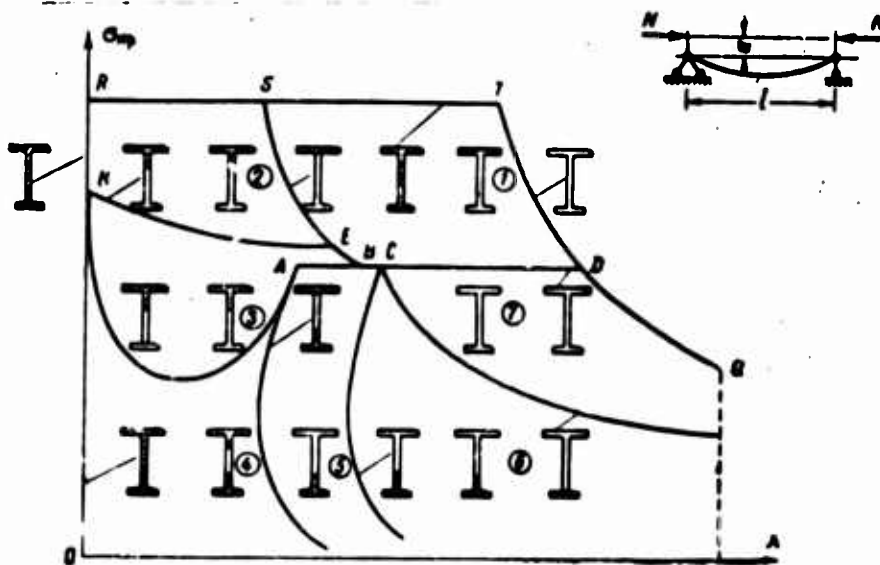


Fig. 10

ly compressed bar, in which the stresses over the entire cross section are equal to  $\sigma_{tp} = \sigma_r$ . Curvilinear segment  $TQ$  is an Eulerian hyperbola and defines the magnitude of critical stresses on elastic buckling of a centrally compressed bar. Both these segments join in point  $T$  which has the abscissa

$$\lambda_r = \pi \sqrt{E/\sigma_r}. \quad (60)$$

In the general case of an asymmetrical H section the plane of the critical-stresses graph is broken up into 7 regions, each of which corresponds to one of the previously examined critical states. These regions are numbered in Fig. 10 by circled numbers. Inside each region is also depicted the stress diagram in the middle section of the bar, which characterizes the given critical state. The boundary curves separating two adjoining regions  $i$  and

$k$  are constructed on the basis of equations of boundary curves  $\lambda_{ik}$ . Figure 10 also shows stress diagrams corresponding to these boundary curves.

The actual outlines of boundary curves, and the areas occupied by the regions on the critical-stresses curve, vary quite substantially with changes in the shape parameters  $\alpha$  and  $\beta$ . Hence the graph of Fig. 10 is "topological" in character; consequently, it only shows the mutual location of regions, and does not refine their configuration.

Finally, we note that the presence of all the seven regions on the critical-stresses graph is not mandatory for H sections with different values of the shape parameters  $\alpha$  and  $\beta$ . For example, only the first four critical states are possible for a symmetrical H section ( $\beta = 1$ ), for which reason the critical-stresses graph does not show regions 5, 6 and 7.

#### §4. Characteristic Points on the Critical-Stresses Graph

Let us note certain characteristic points of the critical-stresses graph (Fig. 10). The straight-line segment  $AD$  parallel to the abscissa axis divides regions 3 and 5, 1 and 5, 1 and 7. This segment is the locus of points corresponding to critical states when fiber flow starts simultaneously from the direction of the compression as well as tension flange. For the critical tension  $\sigma_{o.t}$  in this case we have obtained Relationship (59).

Analysis of this relationship shows that the straight-line segment  $AD$  comes closer to the abscissa axis as the asymmetry parameter  $\beta$  increases from zero to unity and becomes identical with it in the case of  $\beta = 1$  (symmetrical H beam).

Point  $A$  belongs to the above segment  $AD$  and simultaneously also to the two boundary curves  $\lambda_{34}$  and  $\lambda_{45}$ . It follows from comparison of Formulas (41), (45) and (59) that in this point one must satisfy the condition  $\mu_1 = \mu_2 = \alpha$ . Expressing the abscissa of point  $A$  in terms of flexibility  $\lambda_t$ , we find

$$\lambda_A = \sqrt{\frac{\alpha(\alpha + \beta + 1)}{(1 - \beta)\gamma}} \lambda_t. \quad (61)$$

Similarly, for the abscissa of point  $B$ , in which boundary curve  $\lambda_{12}$  meets straight line  $AD$ , we get

$$\lambda_B = \sqrt{\frac{\alpha(\alpha + 4\beta)(\alpha + \beta + 1)}{(\alpha + \beta)(1 - \beta)\gamma}} \lambda_t. \quad (62)$$

For point  $E$  of intersection between the boundary curves  $\lambda_{12}$  and  $\lambda_{23}$  we can find

$$\mu_1^3 \frac{\mu_1 + 1}{\mu_1 + 1} = \alpha^3 \frac{\alpha + 4\beta}{\alpha + \beta}. \quad (63)$$

Point  $C$  of straight line  $AD$  is the meeting point of boundary curves  $\lambda_{56}$  and  $\lambda_{67}$ . Equating Formulas (55), (56) and (59), we find the abscissa of this point

$$\lambda_c = \sqrt{\frac{\alpha(\alpha + 1\beta)(\alpha + \beta + 1)}{(\alpha + \beta)(1 - \beta)\gamma}} \lambda_r. \quad (64)$$

For point  $D$ , at which segment  $AD$  meets Eulerian hyperbola  $TQ$ , we find

$$\lambda_D = \sqrt{\frac{\alpha + \beta + 1}{1 - \beta}} \lambda_r. \quad (65)$$

Boundary curve  $\lambda_{12}$  meets straight line  $RT$ , parallel to the abscissa axis ( $\sigma_{np} = \sigma_r$ ), in point  $S$  with abscissa

$$\lambda_s = \sqrt{\frac{\alpha(\alpha + 1\beta)}{(\alpha + \beta)\gamma}} \lambda_r. \quad (66)$$

Boundary curves  $\lambda_2$ , and  $\lambda_3$ , meet in point  $K$  on the ordinate axis. Setting  $\lambda = 0$ , we find  $\mu_1 = 0$ , whence we get the critical stress corresponding to point  $K$

$$\sigma_K = \frac{\alpha - \beta + 1}{\alpha + \beta + 1} \sigma_r. \quad (67)$$

## §5. Critical Stresses of Asymmetrical H Beams

The critical-stresses graphs in the present work are constructed for steel bars with  $\sigma_t = 2400 \text{ kg/cm}^2$  and  $E = 2,100,000 \text{ kg/cm}^2$ . The assumed values of relative eccentricity are  $v=0.2; 0.6; 1.2; 2.4; 4.8$ .

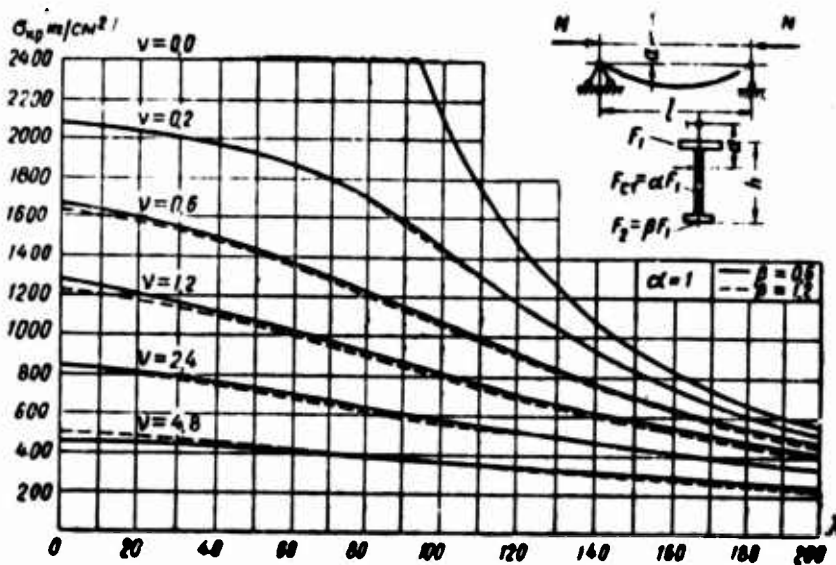


Fig. 11. 1)  $\text{kg/cm}^2$ .

Figure 11 shows a critical-stresses graph for asymmetrical H beams with area ratio parameter  $\alpha = 1$  and asymmetry parameter  $\beta = 0.6$  and  $\beta = 1.2$ . A similar graph for the case of  $\alpha = 2$  is depicted in Fig. 12.

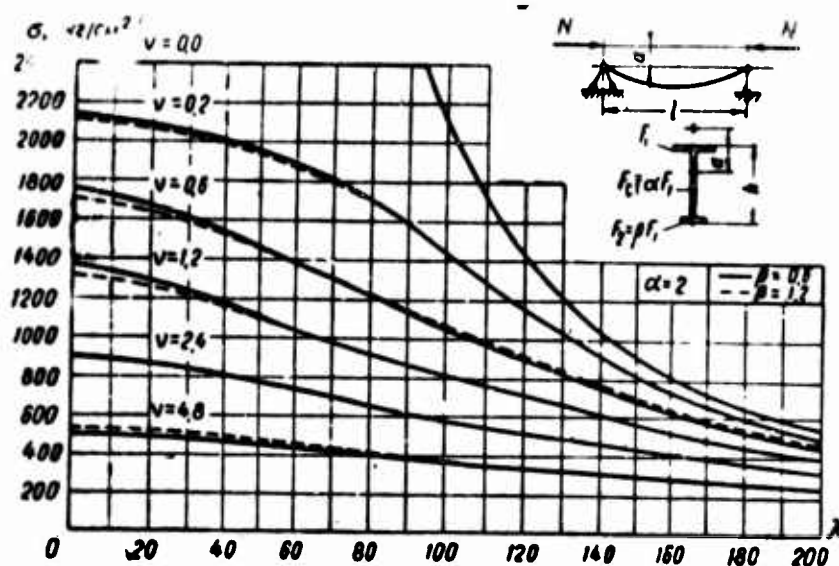


Fig. 12. 1)  $\text{kg/cm}^2$ .

It follows from comparison of results that the critical stresses increase with an increase in the section's area parameter  $\alpha$ . In other words, from among two H sections with the asymmetry parameter  $\beta$  and the same flexibility  $\lambda$  the H section with the thicker web will be the more stable.

The relationship between critical stresses and the asymmetry parameter  $\beta$  is more complex. A certain optimum value of the asymmetry parameter exists for which an H-section bar of a given length taking up a given load has a minimum weight. Analysis of these optimum relationships yields very interesting results, which cannot be presented here.

#### §6. Stability of Bars with a T-Shaped Cross Section, Bending Toward the Web

We consider two independent cases – bending of a T bar toward the web and bending of a T bar toward the flange. The computational relationships for constructing a critical-stresses graph for the T bar will be obtained from previously derived analytical relationships for the more general case of an asymmetrical H section with arbitrary values of parameters  $\alpha$  and  $\beta$ .

We start the study of stability with the case when the T bar bends toward the web. The area ratio parameter is  $\alpha = F_{cw}/F_n$  (Fig. 14), the symmetry parameter  $\beta$  should be equated to zero. Critical states 1, 2, 4, 5, 6 are possible.

In critical state 1 yielding extends to a part of the web thickness. Formula (11) remains applicable for the bar's critical-state flexibility.

In critical state 2 the yielding zone extends to the flange and to the part of the web adjoining it. We retain the notation  $\mu$  for the stressed-state parameter (13). When  $\beta = 0$ ,  $\gamma$  becomes equal to

$$\gamma = \alpha(\alpha + 4)/(\alpha + 1). \quad (68)$$

Setting  $\beta = 0$  in Relationships (21) and (24), we find that in the given case the critical value of the parameter  $\mu$  is

$$\mu_0 = \frac{\alpha}{\alpha + 1} \left[ (\alpha + 2) - \frac{\gamma}{3} (\alpha + 4) \frac{\sigma_{kp}}{\sigma_r - \sigma_{kp}} \right]. \quad (69)$$

On the basis of (28), we find that the critical-state flexibility of the bar is

$$\lambda = \sqrt{\frac{\alpha + 1}{\alpha^2(\alpha + 4)}} \mu_0^{3/2} \lambda_0. \quad (70)$$

The critical state is calculated from Formula (25), the quantity  $K$  taking on the value

$$K = \frac{\gamma}{3} \cdot \frac{\alpha(\alpha + 4)}{\alpha(\alpha + 2) - \mu_0(\alpha + 1)}. \quad (71)$$

Expression (29) for flexibilities corresponding to the boundary between critical states 1 and 2 is obtained in the form

$$\lambda_{12} = \sqrt{\frac{\alpha + 1}{\alpha + 4}} \lambda_0. \quad (72)$$

Substituting the value of  $\gamma$  into Formula (31) and setting  $\beta = 0$ , we get an expression for flexibilities separating critical states 2 and 4

$$\lambda_{24} = \frac{(\alpha + 1)^2}{\sqrt{\alpha^2(\alpha + 4)}} \left( \frac{\sigma_r - \sigma_{kp}}{\sigma_r} \right)^{3/2} \lambda_0. \quad (73)$$

In critical state 4 the plastic compression region extends to the flange and to the part of the web adjoining it. The opposite part of the web is in the plastic tension range.

Transformation of Relationships (37) and (40) yields an expression for the bar's critical-state flexibility

$$\left. \begin{aligned} \lambda &= \sqrt{\frac{\alpha + 1}{\alpha^2(\alpha + 4)}} A^{3/2} \lambda_0; \\ A &= \alpha^2 + 2\alpha - 1 + \left[ 2 - \frac{2}{3} \gamma \alpha (\alpha + 4) \right] \frac{\sigma_{kp}}{\sigma_r} - (\alpha + 1)^2 \left( \frac{\sigma_{kp}}{\sigma_r} \right)^2. \end{aligned} \right\} \quad (74)$$

The critical state for  $\lambda = 0$ ,  $\sigma_{kp} = \sigma_{k,0}$  is found from the condition  $A = 0$ .

We now consider critical state 5. The plastic compression region extends to a part of the flange, with the opposite part of the web in the plastic tension range.

Expression (44) for the bar's critical-state flexibility takes on the form

$$\lambda = \sqrt{\frac{\alpha + 1}{\alpha^2(\alpha + 4)}} \sqrt{6\mu_2(\mu_2 + 2) \rho_2 - 2\mu_2^3} \lambda_0, \quad (75)$$

where on the basis of (42) and (43)

$$\left. \begin{aligned} \mu_2 &= \frac{\sigma_r + \sigma_{kp}}{\sigma_r} (\alpha + 1) - 2; \\ \rho_2 &= \frac{1}{2(\alpha + 1)} \left[ \alpha - \frac{\nu}{3} (\alpha + 4) \frac{\sigma_{kp}}{\sigma_r + \sigma_{kp}} \right]. \end{aligned} \right\} \quad (76)$$

By transforming Formula (45) we get for the flexibilities corresponding to the boundary between critical states 4 and 5

$$\lambda_{45} = \sqrt{\frac{\alpha + 1}{\alpha^2(\alpha + 4)}} \mu_2^{3/2} \lambda_0. \quad (77)$$

It now remains to consider critical state 6. The yielding zone extends to a part of the web at the tension side of the bar.

For the bar's critical-state flexibility we have Formula (51) which for  $\beta=0$  has the form

$$\lambda = \sqrt{\frac{\alpha + 1}{\alpha^2(\alpha + 4)}} \sqrt{\Phi(\rho_2, 1)} \lambda_0. \quad (78)$$

Here  $\Phi(\rho_2, 1)$  is a function of two variables (22) when  $p=\rho_2$ , which is obtained from Formula (76), and  $\beta=1$ .

On the basis of (55) we get for the flexibilities separating critical states 5 and 6

$$\lambda_{56} = \sqrt{\frac{\alpha + 1}{\alpha^2(\alpha + 4)}} \sqrt{\mu_2^3 \frac{\mu_2 + 4}{\mu_2 + 1}} \lambda_0. \quad (79)$$

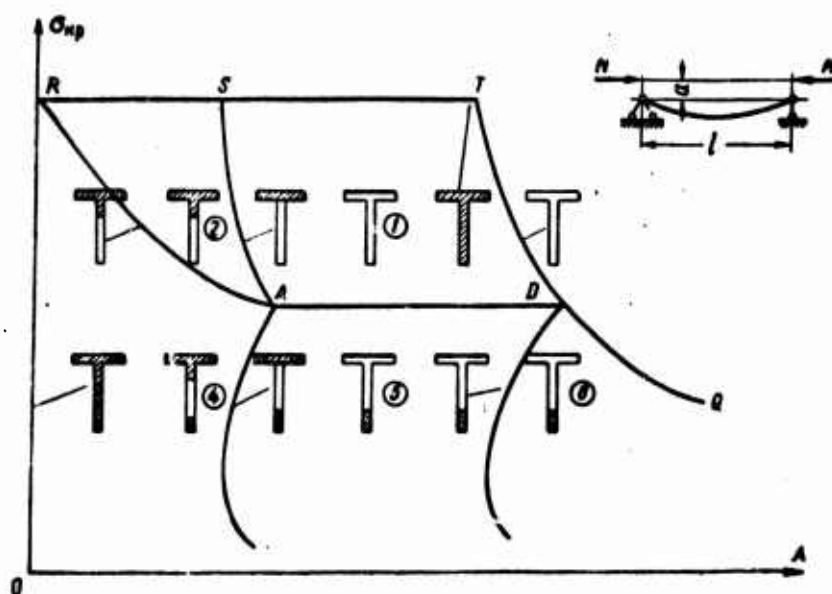


Fig. 13

The distribution of regions corresponding to different critical states of the bar is shown schematically in Fig. 13. The three

points  $A, B$  and  $C$  on straight line  $AD$  (Fig. 10), characteristic for the general case of an asymmetrical H beam, collapse into one point  $A$  with the coordinates

$$\lambda_A = \frac{\alpha+1}{\sqrt{\alpha+4}} \lambda_T, \quad \sigma_{0,T} = \frac{\sigma_T}{\alpha+1}. \quad (80)$$

On the basis of (73) the abscissa of point  $D$  is  $\lambda_D = \sqrt{\alpha+1} \lambda_T$ . Expression (66) for the abscissa of point  $S$ , in which boundary curve  $\lambda_{12}$  meets straight line  $RT(\sigma_{np} = \sigma_T)$  takes on the form  $\lambda_S = \sqrt{(\alpha+1)/(\alpha+4)} \lambda_T$ .

Point  $K$  on the critical-stresses graph (Fig. 10) for the case of a T section becomes coincident with point  $R$ .

Figure 14 depicts a critical-stresses graph constructed for  $\alpha = 0.5$  and  $\alpha = 1.5$ .

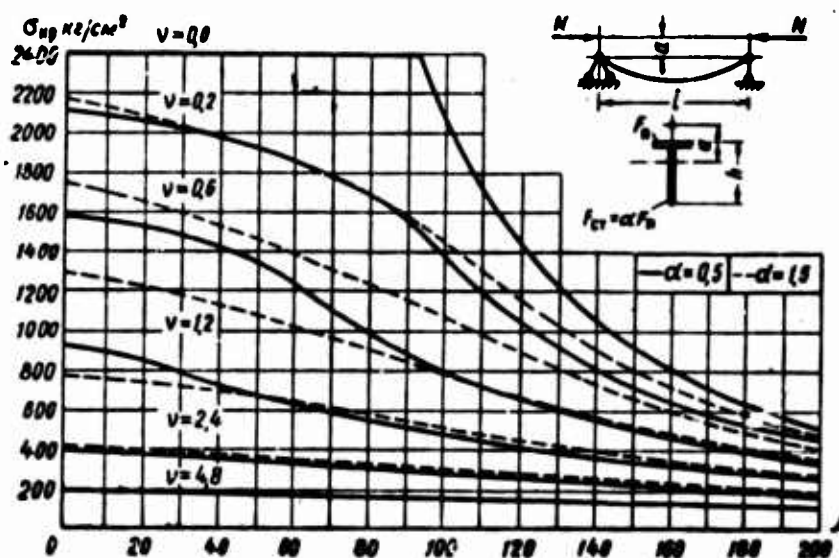


Fig. 14. 1)  $\text{kg/cm}^2$ .

Comparison of the graphs shows that the critical stresses increase with an increasing shape parameter  $\alpha$ .

## §7. Stability of T-Section Bars. Bending Toward the Flange

In the given case the flange is under tension, for which reason the meaning of the notations used for the general case of an asymmetrical H beam changes somewhat. Let  $F_p$  be the flange area and  $F_{st}$  the web area. The shape parameter is the ratio

$$\alpha = F_p/F_{st}. \quad (81)$$

The computational relationships for the case being considered are obtained from the general relationships found in the first section, by replacing in these  $\alpha$  by  $\alpha\beta$  and then letting  $\beta$  go to infinity.

Critical states 2, 3 and 4 are possible when a T bar is bent toward the flange. In critical state 2 yielding extends to a part of the web from the direction of the fiber put into compression by the bending.

After the above transformations, Relationship (24) takes on form

$$\rho = \frac{p_1}{\beta} = \frac{a^2}{2(a+1)} \left( 1 - \frac{\nu}{3} \cdot \frac{a+4}{a+2} \cdot \frac{\sigma_{\kappa p}}{\sigma_T - \sigma_{\kappa p}} \right). \quad (82)$$

For the bar's critical-state flexibility we get from (23)

$$\lambda = \sqrt{\frac{a+1}{a^2(a+4)}} \sqrt{\Phi(p, 1)} \lambda_0, \quad (83)$$

where  $\Phi(p, 1)$  is the result of substituting  $\beta=1$  into Expression (22).

The critical state is calculated from Formula (25), in which

$$K = \frac{\nu}{3} \cdot \frac{a^2}{a+2} \cdot \frac{a+4}{a^2 - 2\rho(a+1)}. \quad (84)$$

The expression for flexibilities corresponding to the boundary between critical states 2 and 3, on the basis of (30) and (31), is written in the form

$$\lambda_{23} = \sqrt{\frac{a+1}{a^2(a+4)}} \sqrt{\mu_1^2 \frac{\mu_1+4}{\mu_1+1}} \lambda_0, \quad (85)$$

where

$$\mu_1 = \frac{\sigma_T - \sigma_{\kappa p}}{\sigma_T} (a+1) - 2. \quad (86)$$

When  $\lambda=0$  we have  $\rho=0$ , and consequently,

$$\sigma_{\kappa,0} = \frac{\sigma_T}{1 + \frac{\nu}{3} \cdot \frac{a+4}{a+2}}. \quad (87)$$

Let us consider critical state 3. The plastic compression region extends to a part of the web. Simultaneously, a part of the web under tension is yielding.

The bar's critical-state flexibility, on the basis of (33), is

$$\lambda = \sqrt{\frac{a+1}{a^2(a+4)}} \sqrt{6\mu_1(\mu_1+2)\rho - 2\mu_1^2} \lambda_0, \quad (88)$$

where  $p$  and  $\mu_1$  are determined by Relationships (82) and (86). For the critical state we have

$$\sigma_{\kappa p} = \left( 1 - \frac{\mu_1+2}{a+1} \right) \sigma_T. \quad (89)$$

In critical state 4 a part of the web is subjected to plastic compression. The flange and the part of the web adjoining it is in plastic tension.

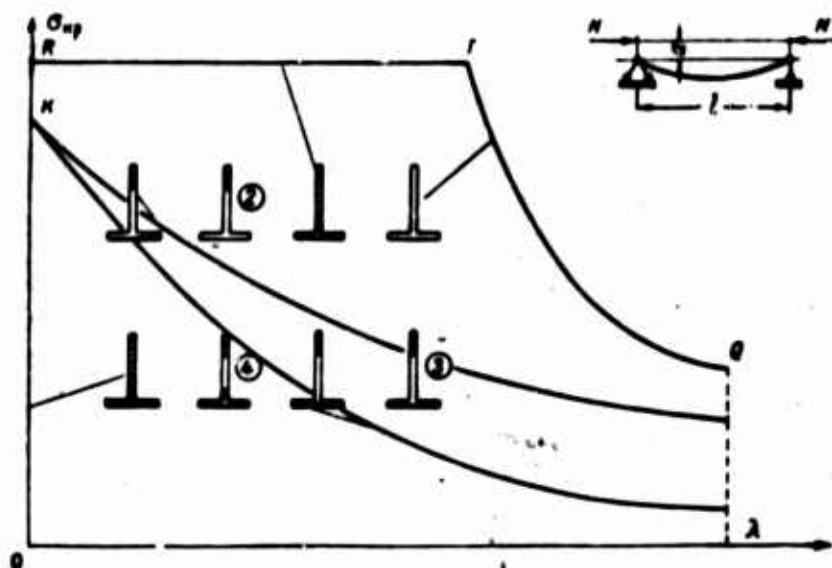


Fig. 15

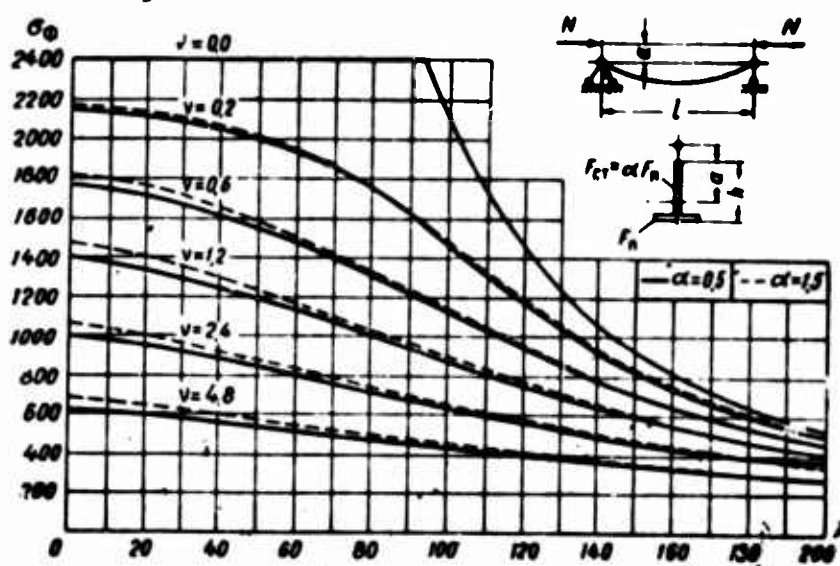


Fig. 16

We use the previously derived Relationships (37) and (40), in which  $\alpha$  should be replaced by  $\alpha\beta$  and then we should set  $\beta = \infty$ . For the bar's critical-state flexibility we find

$$\lambda = \sqrt{\frac{\alpha + 1}{\alpha^2(\alpha + 4)}} A^{1/2} \lambda_0, \quad (90)$$

where

$$A = \alpha^3 + 2\alpha - 1 - \left[ 2 + \frac{2}{3} v \frac{\alpha^2(\alpha + 4)}{\alpha + 2} \right] \frac{\sigma_{ap}}{\sigma_v} - (\alpha + 1)^2 \left( \frac{\sigma_{ap}}{\sigma_v} \right)^2. \quad (91)$$

For the flexibilities separating critical states 3 and 4 we find on the basis of (41)

$$\lambda_{2,4} = \sqrt{\frac{a+1}{a^2(a+4)}} \mu_1' \lambda_1 \quad (92)$$

Critical stress  $\sigma_{k.o.}$  for  $\lambda=0$  is determined from the condition  $A=0$ .

The distribution of regions corresponding to the three possible critical states of the bar is shown schematically in Fig. 5. Boundary curves  $\lambda_2$  and  $\lambda_4$  meet on the ordinate axis in point . On the basis of (67), we have for the corresponding stress  $\sigma_k = \sigma_y(a-1)/(a+1)$ .

Figure 16 displays a critical-stresses graph for  $a=0.5$  and  $a=1.5$ . Comparison of the graphs shows that the critical stresses increase with an increase in shape parameter  $a$ .

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#### Transliterated Symbols

- |    |  |
|----|--|
| 1  | kp = kr = kriticheskiy = critical                  |
| 1  | o = o = osevoy = axial                             |
| 2  | st = st = stenka = web                             |
| 3  | t = t = tekuchest' = yield [point]                 |
| 3  | e = e = eylerovo = Eulerian                        |
| 6  | k.o = k.o = kriticheskoye osevoye = critical axial |
| 10 | o.t = o.t = osevaya tekuchest' = axial yield point |
| 14 | p = p = polka = flange                             |

## STATIC STABILITY OF A STEEL TELEVISION TOWER

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The present paper is concerned with the stability of a steel tower about 315.3 m high. The tower includes an elevator shaft (Fig. 1a) 200 m high and 3 m in diameter, body of the lattice-type prestressed all-welded tower (Fig. 1, branches b), three-story, 13 m high building on top of the tower (Fig. 1c) and a 115.3 m high antenna (Fig. 1d). The antenna is connected to the elevator shaft, which takes up the vertical loads on the antenna. The shape of the body is a hexagonal lattice-type pyramid, narrowing toward the top due to breaks and changes in the slopes of booms; the base width of the tower is 60 m. The tower body has no braces. The main diaphragms connecting the body and the elevator shaft are placed in each junction of the tower's body at each 10-15 m (Fig. 2). The diaphragms are made in the form of hexagonal platforms, whose junctions are connected by an original star-shaped system of flexible tension struts with the elevator-shaft diaphragm (Fig. 3, section I-I). The elevator shaft is an all-welded lattice frame 3x3 m in cross section and with an enclosure in the form of a vertical cylindrical shell from preassembled stiffened steel panels. The upper part of the tower becomes wider at the 187 m elevation to a 6x6 m cross section, and becomes a single entity with the antenna within the limits of the building. The steel structures of the tower body, elevator shaft and the antenna are made from brand 15XCHD steel. The inner tube of the elevator shaft and the latticed body of the tower take up at the 187 m elevation (Fig. 1) a vertical load  $P_1$  of building b and the antenna. In the first approximation, this load is 292 t (the building alone weighs  $P=250$  t). The flexibility of the elevator is much greater than that of the latticed tower, for which reason it becomes necessary to check the static stability of the elevator shaft when subjected at its upper end to the large concentrated force  $P_1$ . At the same time, it is necessary to take into account the elastic junctions between the elevator shaft and the tower body in the form of horizontal diaphragms and the connections between the shaft and the tower in the upper cross section.

We shall examine the buckling of the elevator shaft which takes up the building's load, on the assumption that the junctions of the tower proper do not move, which will make it possible to estimate the relative stability of the elevator shaft. We shall first consider the sufficiently concrete design scheme of the elevator shaft as a bar clamped at its lower end, which has a number of elastic horizontal supports, which simulate the reactions of

the horizontal diaphragms connecting the shaft to the tower (Fig. 2). Then we shall consider the solution of this problem as simulated by a bar on a continuous elastic foundation. Then we shall present a simple solution, which is also sufficiently close to the preceding one, for a bar with one elastic support, reduced to the free end of the bar.

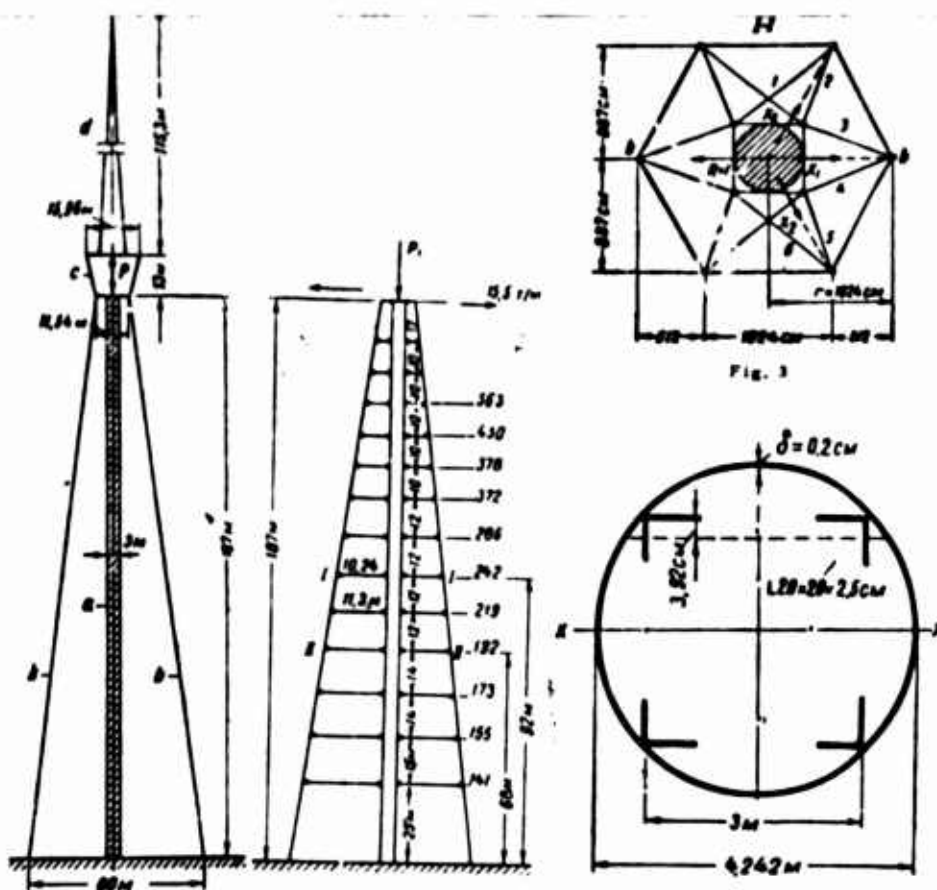


Fig. 1

Fig. 2

Fig. 4

First we solve the problem of stability of the elevator shaft shown in cross section in Fig. 4. The cross-sectional area of the shaft is  $F=641 \text{ cm}^2$ , the section's moment of inertia is  $J_x=13.8 \cdot 10^6 \text{ cm}^4=0.138 \text{ m}^4$ . The area of the middle cross section of the tower body (Fig. 3) is  $F_0=306 \text{ m}^2$ , the moment of inertia of the tower body section relative to an axis parallel to the sides of the hexagon is  $J_{x0}=160 \cdot 10^6 \text{ cm}^4=1.6 \text{ m}^4$ . The radius of inertia of the tower body is 720 cm, and of the elevator shaft is 147 cm. It is necessary to check the stability of the elevator shaft as of a bar, clamped at its lower end and elastically supported at the junctions of the tower body (Fig. 2).

To determine the critical force for the elevator shaft we find the compliance coefficients of the elastic diaphragm supports made of tension struts 1 through 6 in which are produced repulsion forces when force  $R=1$  is applied (Fig. 1). To simplify the calculations, we shall replace each two tension struts which meet in an external junction by an "averaged" tension strut. As a result,

we will get only three forces  $X_2, X_1$ , and  $X_3$  (two unknowns), for which we set up two conditions of equilibrium and compatibility of strains

$$\left. \begin{aligned} 2X_2 \cos 60^\circ + X_1 &= R = 1; \\ \delta_2 &= \delta_1 \cos 60^\circ, \end{aligned} \right\} \quad (1)$$

where  $\delta_1 = \frac{X_1}{EF_d}$ . We find  $X_1 = 0.667$ . For example, for section II-II located at the 68 m elevation, for a minimum diameter  $d = 0.5$  cm we have

$$F_d = \frac{\pi d^2}{4} = \frac{3.14 \cdot 1^2}{4} = 0.785 \cdot 10^{-4} \text{ m}^2.$$

The compliance coefficient of the elastic support

$$m_{el} = \frac{1}{\delta_1} = \frac{EF}{X_1} = \frac{2.1 \cdot 10^7 \cdot 0.785 \cdot 10^{-4}}{0.667 \cdot 13} = 192 \text{ t/m}.$$

All the remaining compliance coefficients for elastic supports shown in Fig. 2 are found similarly.

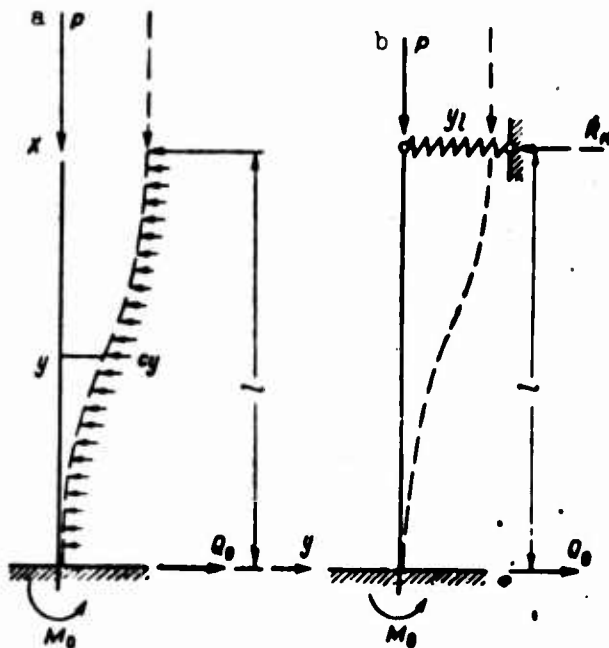


Fig. 5

The problem of stability of the reduced system can be solved in matrix form using a method of Professor A.F. Smirnov [1] on an electronic computer [EC] (3UBM).

If an EC is not available, one should seek an approximate solution, which consists in simulating the tower by a bar on a continuous elastic foundation with constant cross section (Fig. 5a). To determine the stability of this bar, we calculate the deformations, using the equation of the bent axis of a compression-bent

bar on an elastic foundation, presented in Reference [2].

In the given case, making use of the fact that when  $x=0, y_0=0$  and  $\varphi_0=0$ , we have

$$y = \frac{\bar{M}_0}{2a\beta} \operatorname{sh} ax \sin \beta x + \frac{Q_0}{2\beta} \left( \frac{1}{\beta} \operatorname{ch} ax \sin \beta x - \frac{1}{a} \operatorname{sh} ax \cos \beta x \right), \quad (2)$$

where

$$\left. \begin{aligned} a &= \sqrt{0.5(\delta - \gamma)}; \\ \beta &= \sqrt{0.5(\delta + \gamma)}; \\ \gamma &= \frac{P_0}{2EJ}; \quad \delta = \sqrt{\frac{c}{EJ}}; \\ \bar{M}_0 &= \frac{M_0}{EJ}; \quad Q_0 = \frac{Q_0}{EJ}. \end{aligned} \right\} \quad (3)$$

$c$  is the compliance coefficient of the elastic foundation, which is assumed to be constant and equal to

$$c = \frac{m_0}{a} = \frac{192}{14} \approx 14 \text{ t/m}^2.$$

We note that Solution (2) has corresponding to it the case of complex conjugate roots of the characteristic equation of a known 4th order differential equation for a compressed bar on an elastic foundation, when  $\gamma < \delta$ , i.e., when the compliance coefficient of the elastic foundation is large

$$y^{IV} + 2\gamma y'' + \delta^2 y = 0.$$

For this problem this occurs, which is established by preliminary approximate determination of the critical parameter; here it is found that  $\gamma/\delta = 0.217$ .

We now pass on to setting up the characteristic equation. The boundary conditions in the given case are written for  $x=l, y''=0, y'''=0$ .

From (2) we get

$$\left. \begin{aligned} y' &= \frac{\bar{M}_0}{2a\beta} (2a\beta \operatorname{ch} ax \cos \beta x - \gamma \operatorname{sh} ax \sin \beta x) + \\ &+ \frac{Q_0}{2a\beta} (a \operatorname{ch} ax \sin \beta x + \beta \operatorname{sh} ax \cos \beta x); \\ y'' &= \frac{\bar{M}_0}{2a\beta} [(2a^2\beta - \gamma\beta) \operatorname{sh} ax \cos \beta x - \\ &- (2a\beta^2 + \gamma a) \operatorname{ch} ax \sin \beta x] + \\ &+ \frac{Q_0}{2a\beta} [(a^2 - \beta^2) \operatorname{sh} ax \sin \beta x + 2a\beta \operatorname{ch} ax \cos \beta x]. \end{aligned} \right\} \quad (4)$$

Using the above boundary conditions and previously derived expressions, we get the following characteristic equation

$$\left. \begin{aligned} & (2\alpha\beta \operatorname{ch} v \cos v_1 - \gamma \operatorname{sh} v \sin v_1)^2 + \\ & + (2\alpha\beta\gamma - \alpha^2\beta + \alpha\beta^2) \operatorname{sh} v \operatorname{ch} v \sin v_1 \cos v_1 - \\ & - \beta (2\alpha^2\beta - \gamma\beta) \operatorname{sh}^2 v \cos^2 v_1 + \\ & + \alpha (2\alpha\beta^2 + \gamma\alpha) \operatorname{ch}^2 v \sin^2 v_1 = 0, \end{aligned} \right\} \quad (5)$$

where

$$\left. \begin{aligned} v &= \alpha l = l \sqrt{0.5(\delta - \gamma)}; \\ v &= \beta l = l \sqrt{0.5(\delta + \gamma)}. \end{aligned} \right\} \quad (6)$$

If Eq. (5) cannot be satisfied, then the parameter  $\gamma$  is determined by a deformation calculation which, in addition to the axial force, also takes into account the additional wind forces on the upper end. Here the instant at which forces  $M_0, Q_0$  start increasing rapidly is established on the basis of increase in the functions

$$\begin{aligned} & \frac{1}{\alpha\beta} (2\alpha\beta \operatorname{ch} v \cos v_1 - \gamma \operatorname{sh} v \sin v_1), \\ & \frac{1}{\alpha\beta} (\alpha \operatorname{ch} v \sin v_1 + \beta \operatorname{sh} v \sin v_1). \end{aligned}$$

Since exact calculations are quite complicated, we shall approximately estimate the critical force compressing the elevator shaft by reducing all the elastic reactions to a reaction at the free end (Fig. 5b).

The resultant elastic reaction  $R_k$  can be obtained approximately by setting the sums of moments at the built-in end due to all the elastic reactions equal to the moment of the elastic reaction reduced to the free end

$$R_k = m y_l = \frac{1}{P} \left( \sum m_{0i} a_i^3 \right) y_l, \quad (7)$$

where  $a_i$  is the distance of the  $i$ th elastic support to the built-in point, and  $m_{0i}$  is the  $i$ th support compliance coefficient. In accordance with Formula (7) we find for the given tower (Fig. 2)

$$m = \frac{1}{P} \sum m_{0i} a_i^3 = 719 \text{ m/m}.$$

For a bar clamped at the lower end, we have at the upper end at  $x=l$

$$\left. \begin{aligned} & M_k = 0, \quad Q_0 = |R_k|; \\ & R_k = m y_l = \frac{M_0 m l^3}{EJ} \left( \frac{1 - \cos v}{v^3} \right) + \frac{Q_0 m l^3}{EJ} \left( \frac{v - \sin v}{v^3} \right); \\ & v = kl = l \sqrt{\frac{P_{sp}}{EJ}}, \end{aligned} \right\} \quad (8)$$

where the second condition requires that the sum of transverse forces be zero. The above makes it possible to write

$$\left. \begin{aligned} M_0 \cos v + Q_0 l \frac{\sin v}{v} &= 0; \\ M_0 \left[ \frac{ml^3}{EJ} \left( \frac{1 - \cos v}{v^3} \right) \right] + \\ + Q_0 \left[ \frac{ml^3}{EJ} \left( \frac{v - \sin v}{v^3} \right) - 1 \right] &= 0. \end{aligned} \right\} \quad (9)$$

Equating to zero the determinant of coefficients of  $M_0$  and  $Q_0$  in the system of equations (9), we find the following characteristic equation

$$\cos v \left[ \frac{ml^3}{EJ} \left( \frac{v - \sin v}{v^3} \right) - 1 \right] - \left[ \frac{ml^3}{EJ} \left( \frac{1 - \cos v}{v^3} \right) \right] \frac{\sin v}{v} = 0,$$

which, after elementary transformations, is represented in the form

$$\frac{ml^3}{EJ v^3} \left( \cos v - \frac{\sin v}{v} \right) = \cos v. \quad (10)$$

Solving the last equation by successive approximation, we find

$$v = 2.03.$$

From this the approximate value of the critical force

$$P_{kp} = \frac{v^2 EJ}{l^3} = \frac{4.121 EJ}{l^3}. \quad (11)$$

The elevator shaft has a high flexibility ( $\lambda = 196.4$ ).

According to (11), the critical force in this case will be

$$P_{kp} = \frac{4.121 EJ}{l^3} = \frac{4.121 \cdot 2.1 \cdot 10^7 \cdot 0.138}{187^3} = 342 \text{ t}.$$

For the more exact computational scheme (Fig. 5a) we get from deformation calculations that  $P_{kp} = 338 \text{ t}$ , which differs from the approximate value by approximately 1.1%.

Since the acting force (on the assumption that 1/2 of the force is transmitted to the shaft) is 146 t, then we have a sufficient safety factor for stability (2.34). The elevator shaft is entirely stable relative to the body of the tower due to the close spacing of sufficiently stiff diaphragms. Actually  $P_{kr}$ , when the compliance of the tower's junction is taken into account, will be somewhat lower than the value established here.

For an approximate check of the stability of the entire variable cross section tower, we replace it with a constant cross section bar, which is represented by section I-I at the 92 m mark. We have the total moment of inertia of the entire cross section (the tower body and the elevator shaft)

$$J_{\Sigma} = J_{\Sigma 0} + J_{\Sigma} = 1.6 + 0.138 = 1.738 \text{ m}^4.$$

For a bar with free upper end the critical force

$$P_{kp.6} = \frac{\pi^2 EJ_{22}}{4l^2} = \frac{9.87 \cdot 2.1 \cdot 10^7 \cdot 1.738}{4 \cdot 10^4} = 2570 \text{ r,}$$

i.e., the safety factor for stability is substantial (8.8).

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#### Transliterated Symbols

- |    |   |
|----|---|
| 24 | кр = kr = kriticheskiy = critical                       |
| 26 | п = p = [not identified]                                |
| 27 | кр.б = kr.b = kriticheskiy bashni = critical [of] tower |

## CERTAIN QUALITATIVE ESTIMATES OF CRITICAL FORCES IN BAR SYSTEMS

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A large variety of methods is available for obtaining in a large number of cases quite effective solutions for eigenvalues of matrices. At the same time, difficulties which arise in solving equations containing nonlinear functions of the sought parameter have not as yet been overcome.

The present article shows estimates of the smallest positive root (critical value) of equations of the theory of stability of the equilibrium of bar systems, which make it possible to calculate the root with any desired accuracy.

The critical value of the load parameter for which a structure composed of nonthin-walled rectilinear bars and loaded by axial compressive forces loses its stable equilibrium can, as is known, be determined from the condition of existence of a nontrivial solution of the homogeneous system of canonical equations of the method of translations

$$r_{nn}X_n - \sum_{i=1}^{N'} r_{ni}X_i = 0 \quad (n = 1, 2, \dots, N) \quad (1)$$

or in matrix form

$$R\bar{X} = 0.$$

Here  $X_i$  is a translation,  $r_{ni} = r_{ni}(v)$  ( $r_{nn} > 0$ ,  $r_{ni} = -r_{in}$ ) is a reaction arising on translation  $X_i = 1$  in the direction of translation  $X_n$  and  $v$  is a parameter, related to the critical axial compressive force by the relationship

$$v_{kp} = L \sqrt{\frac{P_{kp}}{EJ}}.$$

The prime at the summation sign in Expression (1) shows that a term with number  $i=n$  was dropped.

In Eq. (1) we make the substitution  $\sqrt{r_{nn}}X_n = Z_n$  and solve them for  $Z_n$ . Introducing the notation

$$\frac{r_{ni}}{\sqrt{r_{nn}r_{ii}}} = a_{ni}, \quad (2)$$

we get

$$Z_n = \sum_{i=1}^N a_{ni} Z_i \quad (n = 1, 2, \dots, N). \quad (3)$$

System (3) written in matrix form has the form

$$\bar{Z} = A(v) \bar{Z}. \quad (4)$$

The principal diagonal of symmetrical matrix  $A(v)$  contains zeros, while its other elements are  $a_{ni}$ .

We denote the eigenvalues of matrix  $A(v)$  for an arbitrary, but specific  $v$  by  $\lambda_k$  ( $k=1, 2, \dots, N$ ) and the eigenvectors by  $\bar{Z}_k(Z_1^{(k)}, Z_2^{(k)}, \dots, Z_N^{(k)})$ . Without detriment to generality, we shall assume that the ensemble of all the eigenvectors  $\{\bar{Z}_k\}$  comprises an orthonormal basis of an  $N$ -dimensional space. By definition of an eigenvalue and eigenvector of matrix  $A(v)$

$$\lambda_k \bar{Z}_k = A(v) \bar{Z}_k, \quad (5)$$

or, which is the same thing,

$$\lambda_k Z_n^{(k)} = \sum_{i=1}^N a_{ni} Z_i^{(k)} \quad (n = 1, 2, \dots, N). \quad (6)$$

Let  $v_0$  be the smallest (critical) value of the parameter  $v$ , for which system of equations (1) or its equivalent matrix equation (4) has a nontrivial solution. The following theorems make it possible to establish the upper and lower estimates of the value of  $v_0$ .

**Theorem 1.** For  $v > v_0$  at least one of the eigenvalues of matrix  $A(v)$  is greater than unity.

**Proof.** The quadratic form

$$U = (R\bar{X}, \bar{X}) = (\bar{Z} - A\bar{Z}, \bar{Z}),$$

corresponding to Eq. (4) is represented in the form

$$U = (\bar{Z} - \lambda \bar{Z} + \lambda \bar{Z} - A\bar{Z}, \bar{Z}) = (1 - \lambda) |\bar{Z}|^2 + (\lambda \bar{Z} - A\bar{Z}, \bar{Z}).$$

If we set here  $\lambda = \lambda_k$  and  $\bar{Z} = \bar{Z}_k$  then, on the basis of (5), the second term will become zero and, consequently,

$$U_k = (1 - \lambda_k) |\bar{Z}_k|^2. \quad (7)$$

The values acquired by the quadratic form at orthonormal eigenvectors of the symmetric operator are stationary [1]. But, when  $v > v_0$ , when the bar system is in the unstable state, the matrix of reactions, as is known, is not positive-definite, as a result of which at least one of the stationary values of the corresponding quadratic form is negative. Let  $U_1 < 0$ , then  $1 - \lambda_1 < 0$  and  $\lambda_1 > 1$ . This proves the theorem.

The matrix of reactions can always be transformed so that all

the elements with an even sum of subscripts (with the exception of those in the principal diagonal) would be zero.

Matrix  $A(v)$  in this case for an even  $N$  has the form

$$A(v) = \begin{pmatrix} 0 & a_{12} & 0 & \dots & 0 & a_{1N} \\ a_{21} & 0 & a_{23} & \dots & a_{2,N-1} & 0 \\ 0 & a_{32} & 0 & \dots & 0 & a_{3N} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & a_{N-1,2} & 0 & \dots & 0 & a_{N-1,N} \\ a_{N1} & 0 & a_{N3} & \dots & a_{N,N-1} & 0 \end{pmatrix} \quad (8)$$

For an odd  $N$  the last column and the last row of (8) should be dropped.

**Theorem 2.** For  $v < v_0$  the absolute values of all the eigenvalues of matrix  $A(v)$  which have the form of (8) is less than unity.

**Proof.** Let  $\lambda_1$  be the highest eigenvalue. When  $v < v_0$  the matrix of reactions is positive definite, and hence all the stationary values of quadratic form  $U$  are positive. In particular,  $U_1 > 0$  and, on the basis of (7),  $\lambda_1 < 1$ . It remains to prove that the smallest eigenvalue is  $\lambda_N > -1$ .

If  $\lambda'$  is an eigenvalue and  $Z'(Z'_1, Z'_2, \dots, Z'_N)$  is the corresponding eigenvector, then it is possible to check by direct substitution into Eqs. (3) where, according to (8), for an even  $n+i$ ,  $a_{ni}=0$ , that  $\lambda'' = -\lambda'$  and  $Z''(-Z'_1, Z'_2, \dots, (-1)^N Z'_N)$  is also the eigenvalue and eigenvector of matrix  $A(v)$ . Then when  $\lambda_1 < 1$  the lowest eigenvalue  $\lambda_N = -\lambda_1 > -1$ .

**Theorem 3.** If matrix  $A(v)$  is represented in form (8), then matrix  $A^2(v)$  can be written in the block-diagonal form<sup>1</sup>

$$\widetilde{A^2(v)} = \begin{pmatrix} A_1^2(v) & 0 \\ 0 & A_2^2(v) \end{pmatrix}$$

For an even  $N$  both matrices  $A_1^2(v)$  and  $A_2^2(v)$  have the same eigenvalues. For an odd  $N$  one eigenvalue of the first matrix is zero, while the remaining are also identical with the corresponding eigenvalues of the second matrix.

**Proof.** If  $N$  is odd, then, as follows from the proof of Theorem 2, one of the eigenvalues of matrix  $A(v)$  (middle) should be equal to zero. Adding  $(N+1)$ th zero column and row, we get an even-order matrix which now has two zero eigenvalues, while the remaining eigenvalues are the same as in the starting matrix. As a result of the above, it suffices to prove the theorem only for an even  $N$ .

In matrix  $A(v)$  we rearrange the rows and the columns, placing odd rows and columns first and then even rows and columns. We get

$$\widetilde{A(v)} = \begin{pmatrix} 0 & A_0 \\ A_0^* & 0 \end{pmatrix}$$

Here the asterisk (\*) serves as the transposition sign.

We square  $\tilde{A}(v)$

$$\tilde{A}^2(v) = \begin{vmatrix} A_0 A_0^* & 0 \\ 0 & A_v^* A_v \end{vmatrix}$$

We have two matrices

$$A_1^2(v) = A_0 A_0^* \text{ and } A_2^2(v) = A_v^* A_v$$

The validity of the theorem follows from the fact that the form of the characteristic equation of the product of matrices  $A_0$  and  $A_v^*$ , which supplies the eigenvalues, does not depend on the order of the cofactors [2].

Theorems 1-3 and the identity

$$\text{Sp } A^s = \lambda_1^s + \lambda_2^s + \dots + \lambda_N^s \quad (9)$$

where  $\text{Sp } A^s$  is the  $s$ th degree spur of matrix  $A$  and  $\lambda_1, \lambda_2, \dots, \lambda_N$  are its eigenvalues, make it possible to suggest the following criterion.

Let for a specified  $v$  there exist such a power index  $m=m_1$ , that

$$\text{Sp } A_1^{2m}(v) < 1. \quad (10)$$

Then  $v < v_0$ . If, however, starting with some  $m=m_2$ ,  $\text{Sp } A_1^{2m}(v)$  increases with an increase in  $m$ , then  $v > v_0$ .

In fact, if  $\text{Sp } A_1^{2m_1}(v) < 1$ , then all the eigenvalues of matrix  $A_1(v)$  do not exceed unity by virtue of (9). According to Theorem 1,  $v < v_0$ . An increase in  $\text{Sp } A_1^{2m}(v)$  with an increase in  $m$  points to the fact that at least one of the terms in the right-hand side of (9) is greater than unity. This is possible only when  $v > v_0$ .

Corollary. If at least one of the elements of matrix  $A_1^{2m}(v)$ , for a specified  $v$ , exceeds unity, then  $v > v_0$ .

In fact, in calculating the spurs of degrees of matrix  $A_1^2(v)$  the latter - in this case - can reach a value as high as desired, which shows that the second part of the criterion is satisfied.

In conclusion, we present simpler, although more rough estimates.

Theorem 4. If

$$\max_{n,l} \left| \frac{r_{nl}}{\sqrt{r_{nn} r_{ll}}} \right| > 1, \quad (11)$$

then  $v > v_0$ , and if

$$(N-1) \max_{n,l} \left| \frac{r_{nl}}{\sqrt{r_{nn} r_{ll}}} \right| < 1, \quad (12)$$

then  $v < v_0$ .

Proof. The validity of the first assertion follows from the just formulated corollary. To prove the second assertion of the theorem, we scalarly multiply both parts of Eq. (5) by vector  $\bar{z}_k$ . We get

$$\lambda_k \sum_{n=1}^N [Z_n^{(k)}]^2 = \sum_{n=1}^N \left[ \sum_{i=1}^{N'} a_{ni} Z_i^{(k)} \right] Z_n^{(k)}. \quad (13)$$

We estimate the absolute magnitude of the right-hand side of Expression (13), using for this purpose the triangle inequality and the inequality of [3].

$$\left( \sum_{n=1}^N |Z_n| \right)^2 < N \sum_{n=1}^N Z_n^2.$$

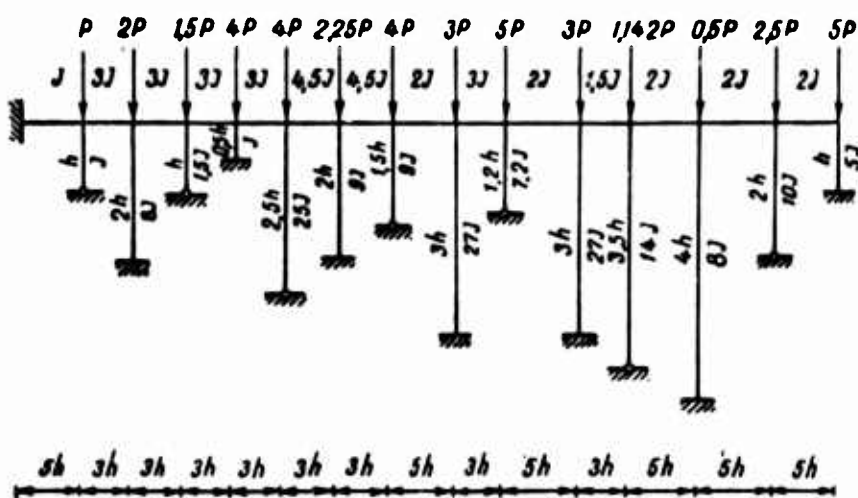


Fig. 1

Dropping the subscript  $k$  in (13), we have

$$\begin{aligned} \left| \sum_{n=1}^N \left[ \sum_{i=1}^{N'} a_{ni} Z_i \right] Z_n \right| &< \max_{n,i} |a_{ni}| \sum_{n=1}^N \sum_{i=1}^{N'} |Z_i| |Z_n| = \\ &= \max_{n,i} |a_{ni}| \left[ \left( \sum_{n=1}^N |Z_n| \right)^2 - \sum_{n=1}^N Z_n^2 \right] < (N-1) \max_{n,i} |a_{ni}| \sum_{n=1}^N Z_n^2. \end{aligned} \quad (14)$$

Equating (13) and (14) and substituting the value of  $a_{ni}$  from (2), we get an estimate of the absolute magnitude of eigenvalue  $\lambda_k$

$$|\lambda_k| < (N-1) \max_{n,i} \left| \frac{r_{ni}}{\sqrt{r_{nn} r_{ii}}} \right|. \quad (15)$$

It follows from (12) and (15) that for any  $k$   $\lambda_k < 1$ , and this, by virtue of Theorem 1, is possible only for  $v < v_0$ .

It is easy to prove that in a number of cases Estimate (12) can be strengthened, i.e., if matrix  $A(v)$  has the form (8), then  $N-1$  is replaced by  $N/2$  for an even  $N$  and by  $\frac{N+1}{2}$  for an odd  $N$ ; for a Jacobian matrix of reactions  $N-1$  should be replaced by the number 2.

Example.<sup>2</sup> Let us consider the stability of the equilibrium of the frame depicted in Fig. 1. The canonical equations of the method of translations have the form

$$-r_{n,n-1}X_{n-1} + r_{nn}X_n - r_{n,n+1}X_{n+1} = 0,$$

where

$$\begin{aligned} r_{11} &= 3\varphi_1(v) + 4,8; & r_{22} &= 16\varphi_1(v) + 8; & r_{33} &= 4,5\varphi_1(v) + 8; \\ r_{44} &= 8\varphi_2(v) + 8; & r_{55} &= 30\varphi_1(v) + 10; & r_{66} &= 18\varphi_2(v) + 12; \\ r_{77} &= 18\varphi_1(v) + 7,6; & r_{88} &= 36\varphi_2(v) + 5,6; & r_{99} &= 18\varphi_1(v) + 5,6; \\ r_{10,10} &= 3,6\varphi_2(v) + 3,6; & r_{11,11} &= 12\varphi_1(v) + 3,6; & r_{12,12} &= 8\varphi_2(v) + 3,2; \\ r_{13,13} &= 15\varphi_1(v) + 3,2; & r_{14,14} &= 20\varphi_2(v) + 1,6; \\ r_{15} &= r_{25} = r_{35} = r_{45} = r_{55} = 2; & r_{66} &= r_{67} = 3; \\ r_{78} &= r_{9,10} = r_{11,12} = r_{12,13} = r_{13,14} = 0,8; & r_{10,11} &= 1. \end{aligned}$$

$$v = L \sqrt{\frac{P}{EJ}}.$$

Functions  $\varphi_1(v)$  and  $\varphi_2(v)$  are tabulated [5].

Since when  $v > 3,33$ ,  $r_{13,13} = 15\varphi_1(v) + 3,2 < 0$ , then  $v_{\text{up}} = v_0 < 3,33$ . Estimates (11) and (12) make it possible to point to a sufficiently narrow isolation range of the sought parameter  $v_0$ :

$$\begin{aligned} \text{for } v = 3,32 \quad \max_n \left| \frac{r_{n,n+1}}{\sqrt{r_{nn}r_{n+1,n+1}}} \right| &= \frac{2r_{12,13}}{\sqrt{r_{12,12}r_{13,13}}} = 0,277 < 1, \\ \text{while for } v = 3,32 \quad \max_n \left| \frac{r_{n,n+1}}{\sqrt{r_{nn}r_{n+1,n+1}}} \right| &= \frac{r_{12,13}}{\sqrt{r_{12,12}r_{13,13}}} = 1,57 > 1. \end{aligned}$$

Thus

$$3,3 < v_0 < 3,32.$$

The isolation range can be further narrowed down by using Inequality (10). Writing

$$\text{Sp } A_1^2 = \sum_{n=1}^{N-1} \frac{r_{n,n+1}^2}{r_{nn}r_{n+1,n+1}}, \quad (16)$$

$$\text{Sp } A_1^4 = \sum_{n=1}^{N-1} \left( \frac{r_{n,n+1}^2}{r_{nn}r_{n+1,n+1}} \right)^2 + 2 \sum_{n=1}^{N-2} \frac{r_{n,n+1}^2 r_{n+1,n+2}^2}{r_{nn}r_{n+1,n+1}r_{n+2,n+2}}, \quad (17)$$

we get from (17)  $\text{Sp } A_1^4(3,31) = 0,469 < 1$ , i.e.,  $3,31 < v_0 < 3,32$ .

The above estimate is in agreement with that obtained by R.R. Matevosyan [4].

Remark. Since  $\text{Sp} A_1^2(3,32) = 4.631$  is smaller than  $\text{Sp} A_1^2(3,32) = 16.243$ , then by virtue of the previously proven criterion,  $3.32 < v_0$ .

The above method is also suitable for finding the natural frequencies of free vibrations.

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## Footnotes

- 30 <sup>1</sup>Matrix  $A_1^2(v)$  is formed from elements of matrix  $A^1(v)$ , lying on the intersection of odd rows and columns, while matrix  $A_2^2(v)$  is composed of elements lying on the intersection of even rows and columns. The remaining elements of matrix  $A^1(v)$  are zeros.
- 33 <sup>2</sup>This example is taken from [4].

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## Transliterated Symbols

- 28  $\kappa p = \kappa r = \text{kriticheskiy} = \text{critical}$

## IMPACT IN SYSTEMS WITH DISTRIBUTED PARAMETERS

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The ever-increasing speeds at which presently used mechanisms are designed to operate have increased the timeliness of the question of strength attendant to impact loads, and primarily of contact strength. As is known [1], solving the problem of impact without considering local crumpling (according to St. Venant) yields incorrect results, for which reason in present-day consideration of the impact process account is taken of the relationship between the contact force and the local deformation, but this relationship is established under static-load conditions. Two approaches to this problem exist.

1) In solving the problem of impact of colliding spheres H. Hertz has used a static relationship in the domain of inverse transforms. The same relationship was used by S.P. Timoshenko [2] in considering impact applied to a simply supported beam.

2) N.A. Kil'chevskiy [3] used operational calculus for considering the static relationship in the domain of transforms. It should be emphasized here that when the inertia terms are not considered the transforms of Lamé equations with zero initial conditions become identical with Lamé's static equations, and this corresponds to the problem of suddenly applied contact force.

Since the relationship between the contact force and contact deformation is in general nonlinear, then these two approaches to the solution are not mutually equivalent. It is, however, of substance that both solutions should become identical in the linear case, and their results can be compared with those obtained experimentally [4] using concentrated masses.

The present paper considers the solution of certain problems stated in both of the above manners. Here we use the following general expression for the relationship between the contact force and deformation [5]:

$$Q = c_k y_k^{\beta}$$

where  $Q$  is the contact force;

$y_k$  is the contact deformation;

$\beta$  is the static nonlinearity index, determined either theoretically or experimentally, here, according to I.Ya.

Shtayerman [6], in the case of the so-called "closer contact"  $1 - \beta < \frac{3}{2}$ .

When  $\beta < 1$  Expression (1) can be used for determining the maximum force attendant to appearance of plastic deformations.

We shall first consider the solution due to N.A. Kil'chevskiy, since the use of operational calculus makes it possible to easily isolate a class of elastic systems with common impact properties; here the system subjected to the impact will be called passive. For it we shall consider local crumpling as well as the over-all deformation, which we shall by convention call the deflection of the system. Only the local crumpling will be considered in the impacting body.

### §1. Solution of the Problem due to N.A. Kil'chevskiy

We write the equation of motion of the impacting mass  $M$  in the form

$$M\ddot{z} = -Q(t) + Q_0, \quad (2)$$

where  $z$  is the absolute displacement of the mass;  $Q(t)$  is the impact force;  $Q_0 = KMg$  is a constant force applied to mass  $M$ ;  $K$  is the static load coefficient.

The Laplace-Carson transform of Eq. (2) has the form

$$Q^*(p) = -Mp^2 z^*(p) + Mpv + Q_0, \quad (3)$$

where  $Q(t) \leftrightarrow Q^*(p)$ ;  $z(t) \leftrightarrow z^*(p)$ ;  $t \leftrightarrow p$ ;  $v$  is the impact velocity; here in this chapter we shall set  $Q_0 = 0$ .

It follows from geometrical considerations that

$$z^*(p) = y_k^*(p) + y_c^*(p), \quad (4)$$

where  $y_k^*$  is the transform of the contact deformation, while  $y_c^*$  is the transform of the deflection of the passive system at the point of impact.

According to [3], we get  $y_k^*$  from Expression (1). Since the passive system is, in the majority of cases, a system with distributed parameters, then the transform of its deflection is represented in the form (see Section 3)

$$y_c^* = \frac{Q^*}{c} \tilde{\psi}^*(p); \quad \tilde{\psi}^*(p) = p^{-\alpha} \sum_{l=0}^{\infty} h_l (a_0 p)^{-\alpha l}; \quad h_0 = 1; \operatorname{Re} p > 0; \quad |p| \rightarrow \infty, \quad (5)$$

where  $\frac{1}{c} \tilde{\psi}^*(p)$  is the transform of the deflection of the passive system due to a suddenly applied unit force; here the representation of Formula (5) for  $\tilde{\psi}^*(p)$  in the vicinity of an infinitely removed point embraces quite a wide class of functions.

Passing in Expressions (5) to the dimensionless time  $\tau_0 = \frac{t}{a_0} \div$   
 $\div s_0 = a_0 p$ , we get

$$y_c^* = \frac{Q^*}{c_0} \tilde{\psi}_0^*(s_0); \quad c_0 = ca_0^{-\alpha}; \quad \tilde{\psi}_0^*(s_0) = s_0^{-\alpha} \sum_{i=0}^{\infty} h_i s_0^{-i}. \quad (6)$$

It will be shown below that a solution in the form of (5) and (6) embraces a wide class of passive systems grouped according to the index  $\alpha$ .

Substituting Expressions (4), (1) and (6) into Eq. (3), we get an equation for the transform of the impact force

$$Q^*(s_0) = -\frac{M}{a_0^2} s_0^2 \left[ Q^* \frac{1}{c_0} + \frac{Q^*}{c_0} \tilde{\psi}_0^*(s_0) \right] + \frac{vM}{a_0} s_0,$$

which we rewrite in the form

$$Q^* \frac{1}{a_0^2 c_0} s_0^2 \left[ 1 + \left( \frac{a_0^2 c_0^{\frac{1}{\alpha}}}{M} \cdot \frac{1}{s_0^2} + c_0^{\frac{1}{\alpha}} \tilde{\psi}_0^* \right) Q^* \frac{s_0^{2-\alpha}}{c_0} \right] = \frac{vM}{a_0} s_0.$$

From this the transform of the impact force can be represented in the form of an infinite chain fraction

$$\left. \begin{aligned} Q^*(s_0) &= \bar{E}_0 \frac{1}{s_0^2 (1+r_1)^{\beta}}; \quad \bar{E}_0 = c_0 (va_0)^{\beta}; \\ r_1 &= \frac{r_0}{(1+r_1)^{\beta-1}}; \quad r_0 = a_0 \left( \frac{b_0}{s_0^2+1} + \frac{\tilde{\psi}_0^*}{s_0^{\alpha-1}} \right); \quad a_0 = \frac{c_0}{c_0} (va_0)^{\beta-1}; \\ b_0 &= \frac{c_0 a_0^2}{M} = \frac{ca_0^{2-\alpha}}{M}. \end{aligned} \right\} \quad (7)$$

Let us consider the expansion of Fraction (7) in powers of the small parameter  $r_0$ , which corresponds to  $|s_0| \rightarrow \infty$ ;  $\text{Re } s_0 > 0$  ( $\tau_0 \rightarrow 0$ ). In this case, the problem reduces to calculating the Burman-Lagrange series [8] for the function

$$D = \frac{1}{(1+r_1)^{\beta}} = d_0 + d_1 r_1 + d_2 r_1^2 + \dots; \quad r_1 = r_0 (1+r_1)^{\beta-1}; \quad r_0 \rightarrow 0. \quad (8)$$

which is feasible, since  $r_0$  has a first-order null in  $r_1=0$ .

According to [8], we get for our case

$$d_0 = 1; \quad d_j = \frac{1}{j!} \lim_{r_1 \rightarrow 0} \frac{d^{j-1}}{dr_1^{j-1}} \left\{ D'(r_1) \cdot \frac{r_1^j}{r_1^j(r_1)} \right\} = (-1)^j \frac{C_{\beta}^{j+1}}{j+1}; \quad j=1, 2, \dots, \quad (9)$$

where  $C_m^0 = 1$ ;  $C_m^n = \frac{m(m-1)\dots(m-n+1)}{n!}$  for any  $m$ ; here  $\beta=1$   $d_j = (-1)^j$ .

The numerical values of coefficients  $d_j$  coincide with those

obtained in another way in [5] for  $\beta = \frac{3}{2}$ . On the other hand, [3] does not have a general approach to calculating the coefficients  $a_j$ , which has made it difficult to use the solution obtained in [3] for the particular case of  $\beta = \frac{3}{2}$ , and has resulted in errors calculating the coefficients of the expansion.

The quantities  $r_\beta^j$  of Series (8) are proportional to  $a_\beta^j$ ; here the convergence of the series improves when  $a_\beta < 1$ , which corresponds to  $c_k < c_0$ . On the other hand, in the case of a large contact stiffness ( $c_k > c_0$ ), which is of particular practical importance,  $a_\beta > 1$ . For this reason it is convenient to change to a new operational variable  $s$  and new dimensionless time  $\tau$ :

$$s = a_1 p; \tau = \frac{t}{a_1}; s = as_0; \tau = \frac{\tau_0}{a}; a_1 = aa_0, \quad (10)$$

where the value of  $a$  is determined from the condition of improving the convergence of Series (8). In this case, instead of Expression (7) for  $a_\beta$  and  $b_0$ , we get

$$a = \frac{a_1}{a_0} = a_\beta^{-\frac{1}{1+\beta-1}}; b = b_0 a^{2-\alpha} = \frac{ca_1^{2-\alpha}}{M}; a_1 = \sqrt[1+\beta-1]{c_k^{-1} c_0^{1-\beta}}. \quad (11)$$

Here when  $a_\beta > 1$   $a < (c_k > c_0)$ , since it was assumed that  $1+\beta > 1$ . It follows from Expressions (11) that the coefficient  $a$  characterizes the relative stiffness of the passive system, while the coefficient  $b$  describes the relative magnitude of the impacting mass  $M$ .

For the new operational variable  $s$  we get instead of Expressions (6)

$$y_i^* = \frac{Q^*}{c_1} \tilde{\psi}_i^*(s); c_1 = ca_1^{-\alpha}; \tilde{\psi}_i^*(s) = a^{-\alpha} \tilde{\psi}_0^* \left( \frac{s}{a} \right) = s^{-\alpha} \sum_{l=0}^{\infty} h_l a^{l\alpha} s^{-l\alpha}; h_0 = 1. \quad (12)$$

Introducing, analogously to Expression (6), the representation

$$\tilde{\psi}_l^m(s) = s^{-\alpha m} \sum_{l=0}^{\infty} h_{l,m} a^{l\alpha} s^{-l\alpha}; h_{0,m} = 1 \quad (13)$$

and substituting into Expressions (7) and (8) instead of  $r_\beta(s_0)$  from (7) the quantity  $r_\beta = r_\beta(s)$ , we get for the transform of the impact force  $Q^*(s)$  the expansion

$$a < 1; Q^*(s) = E_Q \left[ \frac{1}{s^\beta} + \sum_{l=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{d_l a^{l\alpha} b^m C_l^m h_{l,l-m}}{s^{l(\alpha+\beta-1)+m(2-\alpha)+l\alpha+\beta}} \right]; \\ E_Q = c_k (va_1)^\beta, \quad (14)$$

whose inverse transform, taking into account the known formula for the transformation  $s^{-\nu}$  [7], has the form

$$a < 1; Q(\tau) = E_Q \bar{Q}(\tau); E_Q = c_k (va_1)^b;$$

$$\bar{Q}(\tau) = \frac{\tau^2}{\Gamma(1+\beta)} + \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{d_j a^{h_i} b^m c_j^m h_{i,j-m} \tau^{j(i+\beta-1)+m(2-a)+h_i+\beta}}{\Gamma(j(a+\beta-1)+m(2-a)+h_i+\beta+1)}, \quad (15)$$

where  $\bar{Q}(\tau)$  is a dimensionless impact force.

Since Expansion (14) for the transform of the impact force converges in the vicinity of an infinitely removed point in the right half-plane, then Expansion (15) for the inverse transform of the impact force also converges, with the convergence improving with a reduction in  $a, b < 1$ . As follows from (9) and from the expression of formulas for  $h_{i,j,m}$  obtained for specific problems, Series (15) is sign-variable with respect to  $i$  and  $j$ , i.e., the Leibnitz attribute for sign-variable series can be used for evaluating it.

It follows from Series (15) that the main part of the impact force will be obtained by setting in  $\bar{Q}(\tau)$   $a=b=0$ , which corresponds to quite large values of  $c_k$  and  $M$ . In the linear case ( $\beta=1$ ) the transform of the impact force and its principal part ( $a=0$ ) can be represented in a form analogous to Formula (7):

$$Q^*(s) = E_Q \bar{Q}^*(s); \bar{Q}^*(s) = \frac{1}{s \left[ 1 + \frac{b}{s^2} + \bar{\Psi}_1^*(s) \right]};$$

$$\bar{Q}^*(s) \Big|_{a=0} = \frac{1}{s \left( 1 + \frac{b}{s^2} + \frac{1}{s^2} \right)}. \quad (16)$$

It follows from Formulas (5) and (13) that the case of  $a=0$  has corresponding to it the following transform of the deflection of the passive system:

$$a=0; y_c^* = \frac{Q^*}{c} p^{-2}; p \rightarrow t. \quad (17)$$

Consequently, the case of  $a=0$  should be regarded as a case corresponding to limiting expression (17) for the deflection of the passive system when  $c_k \gg c_0$ . Here the constant  $a_1$  (11) in the formula for the dimensionless time  $\tau = \frac{t}{a_1}$  (10) remains finite.

## §2. Solution of the Problem According to Hertz-Timoshenko

Let us consider in this section the process of repeated impacts of the mass  $M$  on a passive system, with each collision preceded by a time of separate motion. Each of the cycles (consisting of separate motion and impact) will be assigned a number, with the subscripts for separate motion placed in parentheses. The start of the  $v$ th impact is denoted by  $T_{v0}$ , its termination is denoted by  $T_{v1}$ . The problem will be considered in dimensionless variables  $\tau, \bar{Q}(\tau)$ , introducing dimensionless displacements  $\bar{A}$  ( $A$  being the dimensional displacement) and  $k$ .

$$\bar{A} = \frac{A}{va_1}; \quad k = \frac{Kga_1}{v} \quad (18)$$

Reducing Eq. (2) and the boundary conditions to the above dimensionless variables, we write them in the form

$$\ddot{\bar{z}}_1 = -b\bar{Q}_1 + k; \quad \tau = T_{10} = 0; \quad \dot{\bar{z}}_{10} = 0; \quad \bar{z}_{10} = 1. \quad (19)$$

Integrating Eq. (19) with respect to  $\tau$ , we get

$$\dot{\bar{z}}_1(\tau) = -b \int_0^\tau \bar{Q}_1(x)(\tau - x) dx + \tau + \frac{k\tau^2}{2}, \quad (20)$$

here  $\bar{z}_1 = \bar{y}_{A1} + \bar{y}_{e1}$ , since the geometric relationships of Formula (4) are valid.

For the dimensionless contact deformation  $\bar{y}_{k1}$ , we will obtain instead of Relationship (1)

$$\bar{y}_{k1}(\tau) = \bar{Q}_1^{\frac{1}{2}}(\tau), \quad (21)$$

and for the dimensionless deflection of the passive system we will get on the basis of Representation (12), using the curl of transformations

$$\bar{y}_{e1}(\tau) = \int_0^\tau \bar{Q}_1(x) \psi(\tau - x) dx, \quad (22)$$

where the kernel of the curl  $\psi(\tau)$  is determined as

$$s\bar{\psi}_1^*(s) = sa^{-1}\bar{\psi}_0^*\left(\frac{s}{a}\right) \div \psi(\tau) = \tau^{a-1}f(\tau); \quad f(0) \neq 0. \quad (23)$$

Formula (23) is valid for  $a > 0$ , since it follows from Expression (5) that  $\bar{y}_e(0) = 0$ .

Substituting Expressions (21) and (22) into (20), we will get a nonlinear integral equation for the impact force  $\bar{Q}_1(\tau)$

$$\bar{Q}_1^{\frac{1}{2}}(\tau) + \int_0^\tau \bar{Q}_1(x) F(\tau - x) dx = \theta_1(\tau); \quad \theta_1(\tau) = \tau + \frac{k\tau^2}{2};$$

$$F(\tau) = \psi(\tau) + b\tau. \quad (24)$$

The first collision terminates when  $\tau = T_{11}$ , when

$$\bar{Q}_1(T_{11}) = 0.$$

In the following separate motion the deflection of the system is written in the form

$$\bar{y}_{(e2)}(\tau) = \int_0^{T_{11}} \bar{Q}_1(x) \psi(\tau - x) dx, \quad (25)$$

and the initial conditions for the mass  $M$  are obtained on the

basis of (20)

$$\tau = T_{11}; \quad \bar{z}_{(20)} = -b \int_0^{T_{11}} \bar{Q}_1(x)(T_{11} - x)dx + T_{11} + \frac{kT_{11}^2}{2};$$

$$\dot{\bar{z}}_{(20)} = -b \int_0^{T_{11}} \bar{Q}_1(x)dx + 1 + kT_{11}. \quad (26)$$

Integrating differential equation (19) for  $\bar{Q}=0$ , we get

$$\bar{z}_{(2)}(\tau) = \bar{z}_{(20)} + \dot{\bar{z}}_{(20)}(\tau - T_{11}) + \frac{k(\tau - T_{11})^2}{2}. \quad (27)$$

It is obvious that the second impact will take place when  $\tau = T_{20}$ , when  $\bar{z}_{(2)}(T_{20}) = \bar{y}_{(2)}(T_{20})$ . The initial conditions for the second impact are determined by the expressions

$$\tau = T_{20}; \quad \bar{z}_{20} = \bar{z}_{(2)}(T_{20}); \quad \dot{\bar{z}}_{(2)} = \dot{\bar{z}}_{(2)}(T_{20}) - \dot{\bar{y}}_{(2)}(T_{20}). \quad (28)$$

The deflection of the passive system before the  $v$ th collision is written in the form

$$\bar{y}_{(v)}(\tau) = \sum_{i=1}^{v-1} \int_{T_{i0}}^{T_{i1}} \bar{Q}_i(x) \psi(\tau - x)dx, \quad (29)$$

and the equations for the force  $\bar{Q}_v(\tau)$  take on the form

$$\left. \begin{aligned} \bar{Q}_v^{\frac{1}{2}}(\tau) + \int_{T_{v0}}^{\tau} \bar{Q}_v(x) F(\tau - x)dx &= 0, (\tau); \\ \bar{Q}_v(\tau) &= \left[ \bar{z}_{v0} - \sum_{i=1}^{v-1} \int_{T_{i0}}^{T_{i1}} \bar{Q}_i(x) \psi(\tau - x)dx \right] + \dot{\bar{z}}_{v0}(\tau - T_{v0}) + \frac{k(\tau - T_{v0})^2}{2}. \end{aligned} \right\} \quad (30)$$

b	1 Расчет на ЭЦВМ				2 Пересчет									
	$\beta=1,5$		$\beta=1,0$		по коэффициенту $b$ , 3 формула (98)				по показателю $\beta$ , формула (91, 93)					
	$\bar{Q}_{\max}$	$\tau_{\max}$	$\bar{Q}_{\max}$	$\tau_{\max}$	$\beta=1,5$	$\Delta$ в %		$\beta=1,0$	$\Delta$ в %		$\Delta$ в %		$\Delta$ в %	
						5	5		5	5	$10^{-3}$ max	$\Delta$ в %	$10^{-3}$ max	$\Delta$ в %
0	0,944	1,715	0,874	1,709	0,944	—	0,874	—	0,935	—1,8	1,966	+15,2	1,495	—12,8
0,125	0,883	1,553	0,818	1,640	0,896	+1,4	0,835	+2,1	0,868	—1,0	1,810	+16,5	1,370	—11,5
0,25	0,829	1,499	0,774	1,450	0,854	+3,0	0,801	+3,4	0,875	—1,6	1,728	+15,3	1,307	—12,8
0,375	0,785	1,419	0,738	1,338	0,817	+4,0	0,770	+4,3	0,773	—1,6	1,614	+13,7	1,221	—14
0,5	0,749	1,346	0,708	1,288	0,784	+4,6	0,748	+4,9	0,787	—1,6	1,448	+16,6	1,188	—11,8

- 1) Calculations on electronic digital computer [EDC] (ЭЦВМ)
- 2) Recalculation
- 3) Using the coefficient  $b$ , Formula (98)
- 4) Using the index  $\beta$ , Formulas (91, 93)
- 5) In.

The above formulas are sufficient for calculating any number

of successive impacts. As can be seen from these formulas, the impact velocity on recurrent impacts is determined by the difference in the velocities of the mass  $M$  and of the passive system (30). Consequently, an increase in the impact forces on recurrent impacts can be expected when the passive system will move toward the mass  $M$ , with the velocity of this system having to exceed the velocity losses of the mass  $M$  during preceding impact collisions in order for this to happen. Here it is obvious that a substantial effect on the intensity of recurrent impacts in real systems will be exerted by damping in the system (for example, in the railroad track bed). We note that consideration of a constant force on impact (the inertia force) results in such a number of impacts, after which the mass is found to be at rest on the passive system.

We now consider a method for calculating the force  $\bar{Q}(\tau)$  from Eqs. (24) and (30).

a)  $\alpha > 1$ . In this case for kernel  $F$  and  $\theta_v$ , the following substantive condition is satisfied:

$$F(0) = 0; \theta_v(T_*) = 0. \quad (31)$$

It is obvious that always  $\bar{Q}(T_*) = 0$ . Replacing the integrals in (24), (30) by a finite sum in the form

$$J_n = \int_{T_*}^{T_*+nh} \varphi(x) dx \approx h \sum_{i=0}^n a_i^{(n)} \varphi_i; \tau_n = T_* + nh, \quad (32)$$

we get a quite simple algorithm for calculating the force  $\bar{Q}_n = \bar{Q}(\tau_n)$ :

$$\alpha > 1; \bar{Q}_0 = 0; \bar{Q}_1 = \theta_v^1; \dots; \bar{Q}_n = (\theta_n - A_n)^2; A_n = h \sum_{i=1}^{n-1} a_i^{(n)} \bar{Q}_i F_{n-i}. \quad (33)$$

On setting up Algorithm (33) in the case of  $n=1$  we have used the trapezoid formula and Condition (31), by virtue of which the sum  $A_n$ , which has replaced the integral in (24) and (30), does not contain the unknown  $\bar{Q}_{vn}$ . As quadrature formula (32) we can use the following:

1) for an even  $n$  - the Simpson formula

$$J_n \approx h \left( \frac{1}{3} \varphi_1 + \frac{4}{3} \varphi_0 + \frac{2}{3} \varphi_2 + \dots + \frac{1}{3} \varphi_n \right), \quad (34)$$

2) for an odd  $n$  - a combination of Simpson's formula with the Newton-Cotes formula for four ordinates (the "3/8 rule")

$$J_n \approx h \left( \frac{1}{3} \varphi_0 + \frac{4}{3} \varphi_1 + \frac{2}{3} \varphi_2 + \dots + \frac{17}{24} \varphi_{n-3} + \frac{9}{8} \varphi_{n-2} + \frac{9}{8} \varphi_{n-1} + \frac{3}{8} \varphi_n \right), \quad (35)$$

with errors proportional to  $h^5 \varphi^{(4)}(\xi); T_* \leq \xi \leq \tau_n$ .

b)  $0 < \alpha \leq 1$ . In this case Condition (31) for  $F(0)$  is not satis-

fied, and for  $0 < \alpha < 1$  quadrature formula (32) should be modified by introducing, near the point  $\tau_n$ , a weight which takes into account the integrated singular point of the kernel  $\psi(\tau)$ . If the function  $\bar{\varphi}(x)$ , remaining after introducing this weight, is replaced at the two extreme intervals by a parabola passing through three points, then the quadrature formula, analogous to the Simpson formula for two intervals, takes on the form:

$$\left. \begin{aligned} \bar{J}_n &= \int_{\tau_{n-2}}^{\tau_n} \frac{\bar{\varphi}(x) dx}{(\tau_n - x)^{1-\alpha}} \approx h \left[ \bar{a}_{n-2}^{(2)} \bar{\varphi}_{n-2} + \bar{a}_{n-1}^{(2)} \bar{\varphi}_{n-1} + \bar{a}_n^{(2)} \bar{\varphi}_n \right]; \\ \bar{a}_{n-2}^{(2)} &= h^{\alpha-1} \frac{2^\alpha \alpha}{(\alpha+1)(\alpha+2)}; \quad \bar{a}_{n-1}^{(2)} = h^{\alpha-1} \frac{2^{\alpha+2}}{(\alpha+1)(\alpha+2)}; \\ \bar{a}_n^{(2)} &= h^{\alpha-1} \frac{2^\alpha (2-\alpha)}{\alpha(\alpha+1)(\alpha+2)}. \end{aligned} \right\} \quad (36)$$

This is precisely the manner in which a quadrature formula with a weight is obtained for one interval, analogous to the trapezoid formula, whose coefficients we denote by  $\bar{a}_{n-1}^{(1)}$  and  $\bar{a}_n^{(1)}$ . For that part of integral  $J_n$  which remains after separating the singular point of the kernel we can use Formulas (34) and (35). Since violation of Condition (31) for  $F(0)$  introduces into the sum replacing integral  $J_n$  an unknown quantity  $\bar{Q}_{vn}$ , then the algorithm for calculating  $\bar{Q}_{vn}$  contains an iteration with respect to  $r$  for  $\beta \neq 1$ :

$$\left. \begin{aligned} 0 < \alpha \leq 1; \quad \bar{Q}_0 = 0; \quad \bar{Q}_{1(r)} &= \left| \frac{\theta_{0,1}}{1 + h \bar{a}_1^{(1)} \bar{F}_0 Q_{1(r-1)}^{1-1/\beta}} \right|^\beta; \\ \bar{Q}_{n(r)} &= \left| \frac{\theta_{0,n} - \bar{A}_n}{1 + h \bar{a}_n^{(2)} \bar{F}_0 Q_{n(r-1)}^{1-1/\beta}} \right|^\beta; \\ \bar{A}_n &= h \left[ \sum_{i=1}^{n-2} \bar{a}_i^{(n-2)} \bar{Q}_{vi} \bar{F}_{n-i} + \bar{a}_{n-2}^{(2)} \bar{Q}_{v, n-2} \bar{F}_2 + \bar{a}_{n-1}^{(2)} \bar{Q}_{v, n-1} \bar{F}_1 \right]; \\ F(\tau) &= \tau^{\alpha-1} \bar{F}(\tau); \quad \bar{F}(\tau) = f(\tau) + b\tau^{2-\alpha}. \end{aligned} \right\} \quad (37)$$

As a zero approximation with respect to  $r$  it is convenient to take in Expression (37)  $\bar{Q}_{vn}(\beta=1)$ , which can be calculated without iteration.

It follows on the basis of the methodology for calculating the impact force, presented in Sections 1 and 2, that the problem of impact may be regarded as solved if: 1) transforms were found for the deflection of the passive system produced by a suddenly applied force and the coefficients of this transform's expansion in terms of the inverse powers of the operational variable (for finding Expansion (15) of the impact force according to N.A. Kil'chevskiy); 2) the inverse transform was found for the kernel of curl (23) for the Hertz-Timoshenko solution [Algorithms (33), (37)].

### §3. Solution for Specific Systems

#### 1. Representation of the impact force in the linear case for elementary systems ( $a=0$ ) using tabulated functions

We consider here the inverse transform of Transform (16) when  $a=0$ , which corresponds to the main part of force (15), which is determined by the principal term of Formula (17) of Expansion (5).

1. Case of  $a=1$ . Impact in the system of a hydraulic shock absorber. For  $a=1$  Formula (17) takes on the form

$$y_c^* = \frac{Q^*}{c} p^{-1}; \quad Q^* = c p y_c^* \div Q(t) = c \dot{y}_c(t), \quad (38)$$

which corresponds to the statement of the heading. Here  $c$  is the damping factor of the shock absorber. It will be shown below that  $a=1$  also corresponds to impact on an infinite plate.

Transform (16) of the impact force for  $a=1$  takes on the form

$$Q^*(s) = E_0 \bar{Q}^*(s); \quad \bar{Q}^*(s) = \frac{s}{s^2 + s + b} \div Q(\tau); \quad s = a_1 p \div \tau = \frac{t}{a_1}, \quad (39)$$

whose inverse, using known inversion formulas [9] is:

$$\bar{Q}(\tau) = \begin{cases} 1 - e^{-\tau}; & b = 0; \\ \frac{2}{\sqrt{1-4b}} e^{-\frac{1}{2}\tau} \operatorname{sh} \sqrt{1-4b} \frac{\tau}{2}; & 0 < b < \frac{1}{4}; \\ \tau e^{-\frac{1}{2}\tau}; & b = \frac{1}{4}; \\ \frac{2}{\sqrt{4b-1}} e^{-\frac{1}{2}\tau} \sin \sqrt{4b-1} \frac{\tau}{2}; & \frac{1}{4} < b. \end{cases} \quad (40)$$

The maximum of impact force  $\bar{Q}(\tau)$  is evaluated quite easily:

$$\left. \begin{aligned} Q_{\max} &= \frac{1}{\sqrt{b}} e^{-\frac{1}{2}\tau_{\max}}; \\ 0 < b < \frac{1}{4}; \operatorname{sh} \sqrt{1-4b} \frac{\tau_{\max}}{2} &= \frac{1 - \sqrt{1-4b}}{2\sqrt{b}}; \\ b &= \frac{1}{4}; \tau_{\max} = 2; \\ \frac{1}{4} < b; \sin \sqrt{4b-1} \frac{\tau_{\max}}{2} &= \frac{\sqrt{4b-1}}{2\sqrt{b}}. \end{aligned} \right\} \quad (41)$$

here, as  $b \rightarrow 0$ ,  $Q_{\max} \rightarrow 1$ , tending to infinity with increasing  $\tau$ . It can be seen from Formula (40) that for  $b \leq \frac{1}{4}$  the impact force tends to zero asymptotically and the impact in the system ends only for  $b > \frac{1}{4}$ .

2 Case of  $\alpha = \frac{3}{2}$  Impact upon an infinite (semi-infinite) beam. Satisfaction of the requirement that  $\alpha = \frac{3}{2}$ , specified in the heading, will be proven below. For  $\alpha = \frac{3}{2}$ , Transform (16) of the impact force takes on the form

$$Q^*(s) = E_0 \bar{Q}^*(s); \bar{Q}^*(s) = -\frac{s}{s^2 + \sqrt{s} + b} \div \bar{Q}(\tau); s = a_1 p \div \tau = \frac{t}{a_1}, \quad (42)$$

whose inverse will be thought applying the Efros transformation [10]

$$D^*(s) \div D(\tau); D^*(\sqrt{s}) \div \int_0^{\sqrt{s}} \frac{e^{-\frac{x^2}{4\tau}}}{\sqrt{\pi\tau}} D(x) dx = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{s}} e^{-x^2} D(2\sqrt{\tau} x) dx \quad (43)$$

to Transform (42)

$$\bar{Q}^*(s) = D^*(\sqrt{s}) = \frac{s}{s^2 + \sqrt{s} + b}; D^*(s) = \frac{s^2}{s^2 + s + b}. \quad (44)$$

The inverse transform  $D^*(s)$  is calculated by residues of the Mellin transform [8]

$$D(\tau) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{e^{s\tau} D^*(s)}{s} ds, \quad (45)$$

in terms of poles of Expression (44)

$$s^2 + s + b = 0. \quad (46)$$

The poles can be determined by solving Eq. (46) in terms of radicals [11]:

$$\begin{aligned} &1) b = 0; \\ &s_1 = 0; s_2 = -1; s_{3,4} = 0.5 \pm i \cdot 0.866; \\ &2) 0 < b < 0.4725; \\ &s_{1,2} = -0.4454 r_0 \pm 0.8903 r_1 \sin \frac{\varphi}{2}; s_{3,4} = \\ &= 0.4454 r_0 \pm i \cdot 0.8908 r_1 \cos \frac{\varphi}{2}; \\ &3) b = 0.4725; \\ &s_{1,2} = -0.63; s_{3,4} = 0.63 \pm i \cdot 0.8908; \\ &4) 0.4725 < b < \infty \\ &s_{1,2} = \sqrt[4]{b} 0.7598 [-R_1 \mp i(R_2 - R_3)]; s_{3,4} = \sqrt[4]{b} 0.7598 \times \\ &\times [R_1 \pm i(R_2 + R_3)]; \\ &r_0 = \sqrt{u+v}; r_1 = \sqrt[4]{u^2+v^2-uv}; \operatorname{tg} \varphi = \sqrt{3} \frac{u-v}{u+v}; \\ &u = \sqrt[3]{1 + \sqrt{1 - (2.1164b)^3}}; \\ &v = \sqrt[3]{1 - \sqrt{1 - (2.1164b)^3}}; R_1 = \sqrt{|\cos \psi|}; \\ &R_{2,3} = \sqrt{\left| \cos \left( \psi \pm \frac{2\pi}{3} \right) \right|}; \cos 3\psi = \frac{3.2476}{b^{3/4}}. \end{aligned} \quad (47)$$

Calculating the residues of Transform (45) in terms of poles defined by Expression (47), we get:

$$\left. \begin{aligned} b > 0; \quad b \neq 0,4725; \quad D(\tau) &= \sum_{k=1}^4 \frac{s_k}{4s_k^3 + 1} e^{s_k \tau}; \\ b = 0; \quad D(\tau) &= \frac{1}{3} \sum_{k=2}^4 \frac{1}{s_k^2} e^{s_k \tau}; \\ b = 0,4725; \quad D(\tau) &= (0,14 - 0,2646 \tau) e^{s_1 \tau} + \sum_{k=2}^4 \frac{s_k}{4s_k^3 + 1} e^{s_k \tau}. \end{aligned} \right\} \quad (48)$$

Substituting Expressions (48) into Formulas (43) of the Efros transformation yields the following expressions for inverse transform (42)

$$\left. \begin{aligned} b > 0; \quad b \neq 0,4725; \quad \bar{Q}(\tau) &= \sum_{k=1}^4 \frac{s_k}{4s_k^3 + 1} w(-is_k \sqrt{\tau}); \\ b = 0; \quad \bar{Q}(\tau) &= \frac{1}{3} \sum_{k=2}^4 \frac{1}{s_k^2} w(-is_k \sqrt{\tau}); \\ b = 0,4725; \quad \bar{Q}(\tau) &= (0,14 + 0,3334\tau) w(-is_1 \sqrt{\tau}) - \\ &- 0,2985 \sqrt{\tau} + \sum_{k=2}^4 \frac{s_k}{4s_k^3 + 1} w(-is_k \sqrt{\tau}), \end{aligned} \right\} \quad (49)$$

where

$$w(z) = e^{-z^2} \left( 1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{x^2} dx \right) - \quad (50)$$

are functions of the complex argument of the probability integral, tabulated in [12].

3. Case of  $\alpha=2$ . Collision impact of two masses. When  $\alpha=2$ , Formula (17) takes on the form

$$y_c^* = \frac{Q^*}{c} p^{-2}; \quad Q^* = cp^2 y_c^* \div Q(t) = c \ddot{y}_c(t), \quad (51)$$

which corresponds to the situation of the heading if  $c=M_2$ .

When  $\alpha=2$  the formulas in the N.A. Kil'chevskiy solution can be simplified by combining in  $r_\beta$  of Formula (7) the term  $b_0 s_0^{-(1+\beta)}$  with the first term of expansion  $\hat{\psi}_0^* s_0^{(1-\beta)}$ , with the result that it is no longer necessary to sum over  $m$  in Expression (15). In the case of collision impact of masses  $M$  and  $M_2$ , when only one term of Expansion (6) is present, Formula (15) takes on the form

$$\left. \begin{aligned} Q(\tau) &= E_Q \bar{Q}(\tau); \quad \bar{Q}(\tau) = \frac{\tau^\beta}{\Gamma(1+\beta)} + \sum_{j=1}^{\infty} d_j \frac{\tau^{(1+\beta)j+\beta}}{\Gamma[(1+\beta)(j+1)]}; \\ \tau &= \frac{t}{a_1}, \end{aligned} \right\} \quad (52)$$

where in Formulas (11) and (15) for  $a_1$  and  $E_Q$  one should set  $a = -\frac{MM_2}{M+M_2}$  due to combining the terms of  $r_\beta$  in (7).

It is quite obvious from Formulas (16) and (52) that when  $\alpha=2$  and  $\beta=1$  we will get a trivial formula for the inverse transform of the impact force

$$\beta=1; \quad Q(\tau) = E_Q \bar{Q}(\tau); \quad \bar{Q}(\tau) = \sin \tau; \quad \bar{Q}_{\max} = 1; \quad \tau_{\max} = \frac{\pi}{2}; \quad (53)$$

corresponding to collision impact of two masses separated by a linear spring.

We now present in our notation the solution for collision impact of two masses according to Hertz with a nonlinearity indicator  $\beta$ , obtained on the basis of the solution in [3]:

$$\left. \begin{aligned} Q &= E_Q \bar{Q}(\tau); \\ \bar{Q}(\tau) &= \left( \frac{1+\beta}{2} \right)^{\frac{\beta}{1+\beta}} \xi^\beta; \quad \xi = \frac{\bar{y}_k}{\bar{y}_{k\max}}; \quad \bar{y}_{k\max} = \left( \frac{1+\beta}{2} \right)^{\frac{1}{1+\beta}}; \\ \tau = \frac{t}{a_1} &= \bar{y}_{k\max} \int_0^\xi \frac{dx}{\sqrt{1-x^{1+\beta}}} = \bar{y}_{k\max} \frac{B_z\left(\frac{1}{1+\beta}, \frac{1}{2}\right)}{1+\beta}; \quad z = \xi^{1+\beta}, \end{aligned} \right\} \quad (54)$$

where  $B_z(p, q)$  is an incomplete beta-function.

In conclusion of problem "a" we note that an attempt to obtain an expression for the impact force in terms of tabulated functions results, even in the most elementary case of  $\alpha = 0$  in quite complex expressions (even when  $\beta = 1$ ) for nontrivial systems, for example, for an infinite beam ( $\alpha = \frac{3}{2}$ ). Consequently, it is reasonable to expect that in more complex systems it is impossible to represent the exact solution of the impact problem in terms of tabulated functions, which means that the methods presented in Sections 1 and 2 do yield results.

*b. Infinite (semi-infinite) beam with linear damping on an elastic Winkler support*

The equation of free vibrations of the beam in this case has the form

$$EJ_0 y_c^{IV} + N_0 \dot{y}_c + u_c y_c + \rho_0 \ddot{y}_c = 0, \quad (55)$$

where  $y_c$  is the beam's deflection;  $EJ_0$  is its flexural rigidity;

$N_b$  is the linear damping coefficient, referred to the beam;  $u_0$  is the elastic modulus of the foundation;  $\rho_b$  is the beam's mass per unit length.

The Laplace-Carson transform of Eq. (55) with zero boundary conditions is

$$EJ_b y_c^{*IV} + (u_c + N_b p + \rho_b p^2) y_c^* = 0; \quad y_c^* \rightarrow y_c; \quad p \rightarrow t. \quad (56)$$

Reducing Eq. (56) to the standard form for static design of beams on an elastic foundation, and using the ready static solution [13], we will get a representation of the deflection beneath a suddenly applied force for a semi-infinite and infinite beam in the form

$$\begin{aligned} y_c^* &= \frac{Q^*}{c} \tilde{\psi}^*(p) = \frac{Q^*}{c_0} \tilde{\psi}_0^*(s_0) = \frac{Q^*}{c_1} \tilde{\psi}_1^*(s); \\ \tilde{\psi}_0^*(s_0) &= (s_0^2 + n_0 s_0 + 1)^{-3/4}; \quad \tilde{\psi}_1^*(s) = (s^2 + ns + a^2)^{-3/4}; \\ c &= \nu \frac{\rho_b}{k_d}; \quad k_A = \sqrt[4]{\frac{\rho_b}{4EJ_b}}; \quad k_c = \sqrt[4]{\frac{u_c}{4EJ_b}}; \\ a_0 &= \sqrt{\frac{\rho_b}{u_c}}; \quad n_0 = \frac{N_b a_0}{\rho_b}; \\ c_0 &= c a_0^{-3/2} = \nu \frac{u_c}{k_c}; \quad c_1 = c a_1^{-3/2}; \quad n = a n_0 = \frac{N_b a_1}{\rho_b}; \\ a &= \frac{a_1}{a_0}. \end{aligned} \quad (57)$$

Here  $c_0$  is the static stiffness of a beam on a Winkler foundation, with  $\nu=2$  for an infinite and  $\nu=\frac{1}{2}$  for a semi-infinite beam.

Expression (57) corresponds to  $a=\frac{3}{2}$  in (6) and (12) and for  $n=a=0$  ( $N_b=u_c=0$ ) yields the beam which was examined in Subsection "a."

Using the known operational relationship [7]:

$$\lim_{\tau \rightarrow \infty} f(\tau) = \lim_{s \rightarrow 0} f^*(s), \quad (58)$$

we shall show that deflection  $y_c$  on sudden application of a constant force tends to the static value for  $\tau \rightarrow \infty$ . On the other hand, for a beam without a foundation ( $a=0$ ) the deflection increases without bounds, i.e., the model of an infinite beam without a foundation can be used only for calculations of the first stage of the impact.

Using Eq. (56) and the already available static solution of [13], we will get the bending moment in an infinite beam beneath the force in terms of a curl of representations, assuming that  $Q(\tau) = E_0 \bar{Q}(\tau)$  is known:

$$\begin{aligned} M_{n\tau}^* &= \frac{1}{4k_1} \cdot \frac{1}{s} Q^*(s) \frac{s}{(s^2 + ns + a^2)^{1/4}} \rightarrow M_{n\tau} = \\ &= \frac{E_0}{4k_1} \int_0^\tau \bar{Q}(x) \frac{m(\tau-x) dx}{(\tau-x)^{1/2}}; \quad k_1 = k_A a_1^{-1/2} \end{aligned} \quad (59)$$

The kernel of curl (59) corresponds to the following inverse transform

$$\frac{s}{(s^2 + ns + a^2)^{1/4}} \div \varphi(\tau) = \tau^{-1/2} m(\tau). \quad (60)$$

On the other hand, the kernel of curl (33) which is needed for solving the problem of impact according to Hertz-Timoshenko corresponds in the case of (57) to

$$\tilde{s}\psi_1(s) = \frac{s}{(s^2 + ns + a^2)^{3/4}} \div \psi(\tau) = \tau^{1/2} f(\tau). \quad (61)$$

The inverse transforms of kernels (60) and (61) are a particular case of an inverse transform which can be calculated applying the rules of operational calculus [7, 9]:

$$\left. \begin{aligned} v > 0; \quad \frac{s}{(s^2 + ns + a^2)^v} \div e^{-\frac{n\tau}{2}} \left(\frac{2\tau}{E}\right)^{v-\frac{1}{2}} \frac{\Gamma(v+\frac{1}{2})}{\Gamma(2v)} J_{v-\frac{1}{2}}(E\tau); \\ E = \sqrt{a^2 - \frac{n^2}{4}}, \end{aligned} \right\} \quad (62)$$

where  $J_v(x)$  is a first-order Bessel function.

The sought kernel (61) is obtainable from Expression (62) for  $v = \frac{3}{4}$ , while kernel (60) can be obtained for  $v = \frac{1}{4}$ . It is seen from Formula (62) that in the case of a large resistance, when  $n \geq 2a$ , the beam's vibrations are aperiodic, since  $J_v$  becomes  $I_v$ .

It can be proven that in the given problem Expansion (15) of the impact force according to Kil'chevskiy will contain a degenerate hypergeometric function due to the presence of three parameters ( $a$ ,  $b$  and  $n$ ). In particular cases, when  $a$  and  $n$  are taken into account separately, Formula (15) retains its previous form. Taking into account Expressions (12), (13) and (57), we get for the coefficients of Expansion (15)

$$\left. \begin{aligned} 1) \quad a \neq 0; \quad n = 0; \quad a = \frac{3}{2}; \quad k=2; \quad h_{l,m} = C_{-3/4m}^l; \quad h_{0m} = 1; \\ 2) \quad a = 0; \quad n \neq 0; \quad a = \frac{3}{2}; \quad k=1; \quad h_{l,m} = C_{-3/4m}^l; \\ h_{0m} = 1; \quad a = n. \end{aligned} \right\} \quad (63)$$

Expression (63) validates our claim that Series (15) is sign-variable with respect to  $i$ .

#### *c. Infinite (semi-infinite) beam on elastic-mass bars*

This computational model of a foundation for steady-state vibrations was considered in [14, 18]. Being sufficiently simple, it makes it possible to clarify the principal effect of the inertia forces of the foundation, which are not taken into account by the Winkler model. Thus, we shall assume that the beam is at rest on unconstrained elastic-mass bars of finite length  $l$ , fastened by their opposite end on an undeformable plate. The equation of

longitudinal vibrations of the bar has the form

$$C^2 \frac{\partial^2 z}{\partial \xi^2} = \frac{\partial^2 z}{\partial t^2}; \quad C = \sqrt{\frac{E_r}{\rho_r}}, \quad (64)$$

where  $z$  is the displacement along the bar;  $\xi$  is the coordinate along a bar with an origin at the upper end;  $E_r, \rho_r, C$  are the elastic modulus, specific weight of the bar and the rate of propagation of compression waves in it.

The transform of Eq. (64) with zero initial conditions is

$$C^2 z^{**} - p^2 z^* = 0. \quad (65)$$

Solving Eq. (65) with load  $\tilde{n}^*(p)$  applied to the upper end of the bar whose lower end is fastened, we get the displacement of the upper end, which is equal to the deflection of the beam  $y_c^*$  in the form

$$z^*|_{\xi=0} = y_c^* = -\frac{\tilde{n}^*(p) C}{E_r \rho} \operatorname{th} \frac{pl}{C}. \quad (66)$$

Using Formula (66) one can obtain a formula for the elastic repulse of bars  $R$ , which acts on the beam

$$\left. \begin{aligned} R &: R^*(p) = \delta \tilde{n}^*(p) = -\frac{\delta E_r}{C} p \operatorname{cth} \frac{pl}{C} y_c^* = \\ &= -\frac{\delta E_r}{C} \left( 1 + 2 \sum_{k=1}^{\infty} e^{-\frac{2plk}{C}} \right) p y_c^*; \\ R &= -\frac{\delta E_r}{C} \dot{y}_c(t); \quad 0 \leq t \leq \frac{2l}{C}; \\ &= -\frac{\delta E_r}{C} \left[ \dot{y}_c(t) + 2 \dot{y}_c \left( t - \frac{2l}{C} \right) \right]; \quad \frac{2l}{C} \leq t \leq \frac{4l}{C}, \end{aligned} \right\} \quad (67)$$

where  $\delta$  is the width of the beam;  $y_c$  is the inverse transform of its deflection; here  $R \rightarrow \frac{\delta E_r}{l} y_c$  as  $t \rightarrow \infty$ .

Consequently, the elastic repulse of the bars is determined by the successive reflection of compression waves from the fastened ends of the bars, which corresponds to introduction of linear damping into the beam and is a result of consideration of the inertia forces of the bars. For  $0 \leq t \leq \frac{2l}{C}$  according to Expression (67) a beam on elastic-mass bars can be examined using as a model problem "b" with damping

$$N_\delta = \frac{\delta E_r}{C} = \delta C \rho_r = u_c \frac{l}{C} = \sqrt{u_c l \delta \rho_r},$$

where  $u_c = \frac{\delta E_r}{l}$  is the static modulus of elasticity of the foundation; here  $N_\delta$ , the beam's damping, increases with an increase in the foundation's elastic modulus and in the mass of the foundation which participates in the vibrations.

Proceeding in the same manner as in problem "b," we will get the transform of the beam's deflection due to a suddenly applied force in the form

$$\left. \begin{aligned} y_s^* &= \frac{Q^*}{c} \tilde{\Psi}^*(s) = \frac{Q^*}{c_0} \tilde{\Psi}_0^*(s_0) = \frac{Q^*}{c_1} \tilde{\Psi}_1^*(s); \\ \tilde{\Psi}_0^*(s_0) &= s_0^{-3/4} (s_0 + \text{cth } H s_0)^{-3/4}; \quad \tilde{\Psi}_1^*(s) = s^{-3/4} \left( s + a \text{cth } \frac{Hs}{a} \right)^{-3/4}; \\ c &= \nu \frac{\rho_0}{k_g}; \quad k_g = \sqrt{\frac{\rho_0}{4EJ_0}}; \quad c_0 = c a_0^{-3/2}; \quad c_1 = c a_1^{-3/2}; \\ a_0 &= \frac{\rho_0}{N_0}; \quad H = \frac{l b \rho_r}{\rho_0}. \end{aligned} \right\} \quad (68)$$

In the case when the impact interval under study is  $t > \frac{2l}{c}$  ( $\tau > \frac{2H}{a}$ ) and the model of problem "b" cannot be used, then in order to take into account the reflected waves (the fact that the elastic-mass layer is finite) one must find the inverse transform of kernel (23) in case (68)

$$s \tilde{\Psi}_1^*(s) = \frac{s}{s^{3/4} \left( s + a \text{cth } \frac{Hs}{a} \right)^{3/4}} \rightarrow \psi(\tau) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{s\tau} ds}{s^{3/4} \left( s + a \text{cth } \frac{Hs}{a} \right)^{3/4}}. \quad (69)$$

It follows from Expression (69) that the singular points in this case will be branching points located in  $s=0$  and in the roots of the equation

$$s + a \text{cth } \frac{Hs}{a} = 0; \quad s = \pm i B_k; \quad k = 0, 1, 2, \dots \quad (70)$$

Aligning the integration contour in Formula (69) with the imaginary axis, we get

$$\begin{aligned} \psi(\tau) &= \frac{1}{\pi \sqrt{2}} \sum_{k=0}^{\infty} \int_{B_k}^{A_{k+1}} \frac{(\sin \tau y - \cos \tau y) dy}{y^{3/4} \left( y - a \text{ctg } \frac{Hy}{a} \right)^{3/4}} + \\ &+ \frac{1}{\pi} \sum_{k=0}^{\infty} \int_{A_k}^{B_k} \frac{\cos \tau y dy}{y^{3/4} \left( a \text{ctg } \frac{Hy}{a} - y \right)^{3/4}}, \end{aligned} \quad (71)$$

where points  $A_k$  correspond to  $\text{ctg } \frac{HA_k}{a} = 1$ ,  $\infty$ .

The integrals obtained above have integrable singular points in  $A_0=0$  and  $B_k$ . These integrals can be evaluated by using the integral

$$J = \int_A^B \frac{e^{i\tau y} dy}{y^{3/4}(y)} = \int_A^B \frac{e^{i\tau y} \varphi(y) dy}{(y-A)^{3/4}}; \quad f(A) = 0. \quad (72)$$

Integrating by parts, we get

$$\left. \begin{aligned} J &= 4(B-A)^{1/4} e^{i\sqrt{B}} \left[ \frac{B-A}{f(B)} \right]^{3/4} - 4 \int_A^B e^{i\sqrt{y}} (y-A)^{1/4} \psi(y) dy; \\ \psi(y) &= \left[ \frac{f(y)}{y-A} \right]^{1/4} \left[ \frac{y-A}{f(y)} + \frac{3}{4} \cdot \frac{f(y) - f'(y)(y-A)}{f^2(y)} \right], \end{aligned} \right\} \quad (73)$$

here  $\psi(y)$  in the vicinity of singular point  $y=A$  can, with a small error, be represented as

$$\psi(y) \approx \frac{1}{|f'(y)|^{3/4}} \left[ 1 - \frac{3}{8} \cdot \frac{f''(y)}{f'(y)} \right]. \quad (74)$$

The integral in (73) no longer has a singular point and can be evaluated by Filon [sic] formulas for rapidly varying integrals [15].

Contour integral (69) can also be evaluated by a quadrature formula for contour integrals on the complex plane, suggested by Salzer [16] and developed in [17]; however, the applicable expressions are not presented due to limited space.

#### d. Infinite plate with linear damping

1. Infinite plate on an elastic Winkler foundation. Using the solutions given in [7] and additionally introducing damping into the plate, we will get a transform of the beam's deflection due to a suddenly applied force in the form

$$\left. \begin{aligned} y_c^* &= \frac{Q^*}{c} \tilde{\psi}^*(p) = \frac{Q^*}{c_0} \tilde{\psi}_0^*(s_0) = \frac{Q^*}{c_1} \tilde{\psi}_1^*(s); \\ \tilde{\psi}_0^*(s_0) &= (s_0^2 + n_0 s_0 + 1)^{-1/2}; \quad \tilde{\psi}_1^*(s) = (s^2 + ns + a^2)^{-1/2}; \\ c &= 8\sqrt{D_n \rho_n}; \quad a_0 = \sqrt{\frac{\rho_n}{u_c}}; \quad c_0 = ca_0^{-1} = 8\sqrt{D_n u_c}; \\ c_1 &= ca_1^{-1}; \quad n_0 = \frac{N_n a_0}{\rho_n}; \quad n = \frac{N_n a_1}{\rho_n}, \end{aligned} \right\} \quad (75)$$

where  $D_n$ ,  $N_n$ ,  $\rho_n$  are the cylindrical flexural rigidity, the damping coefficient and mass of the plate;  $u_c$  is the elastic modulus of the foundation;  $c_0 = 8\sqrt{D_n u_c}$  is the static rigidity of an infinite plate on a Winkler foundation.

Expression (75) corresponds to  $a = 1$  in (6) and (12). The inverse transform of kernel (23) in the case of (75) is calculated according to the rules of operational calculus [7, 9]

$$s\tilde{\psi}_1^*(s) = \frac{s}{\sqrt{s^2 + ns + a^2}} \rightarrow \psi(\tau) = e^{-\frac{n\tau}{2}} J_0(E\tau); \quad E = \sqrt{a^2 - \frac{n^2}{4}}, \quad (76)$$

where  $J_0(x)$  is a first-kind Bessel function.

It follows from Formula (76) that in the case of a large resistance, when  $n > 2a$ , the plate vibrations are aperiodic, since  $J_0$  becomes  $I_0$ .

The coefficients of Expansion (15) of the impact force according to Kil'chevskiy in the given problem have the form

$$\left. \begin{aligned} 1) a \neq 0; n = 0; a = 1; k = 2; h_{i,m} = C_{-\frac{m}{2}}^i; h_{0m} = 1; \\ 2) a = 0; n \neq 0; a = 1; k = 1; h_{i,m} = C_{-\frac{m}{2}}^i; \\ h_{0m} = 1; a = n. \end{aligned} \right\} \quad (77)$$

Formula (77) validates our contention that Series (15) is sign-variable with respect to  $i$ .

2. Infinite plate on an inertialess elastic half-space. The equation of the plate's vibrations has the form

$$D_n \Delta \Delta y_c + N_n \ddot{y}_c + \rho_n \ddot{y}_c = \tilde{Q} - \tilde{R}, \quad (78)$$

where  $\tilde{Q}$  is a distributed load acting on the plate;  
 $\tilde{R}$  is the reaction of the half-space, while the balance of the notation is that of Expression (75).

The Laplace-Carson transform of Eq. (78) with zero initial conditions is

$$D_n \Delta \Delta y_c^* + (\rho_n p^2 + N_n p) y_c^* = \tilde{Q}^* - \tilde{R}^*. \quad (79)$$

Applying to the solution of Eq. (79) for the case of a concentrated force  $Q$  acting on an infinite plate the method due to B.G. Korenev [18], we will get the transform of the deflection in the form of the integral

$$y_c^* = \frac{Q}{2\pi} \int_0^\infty \frac{J_0(\gamma r) \gamma d\gamma}{\frac{1}{C(\gamma)} + D_n \gamma^4 + N_n p + \rho_n p^2}; \quad p \div t. \quad (80)$$

According to [18], for an inertialess elastic half-space we assume

$$\frac{1}{C(\gamma)} = \frac{E_r}{2(1-\nu^2)} \gamma = \frac{\mu_r}{1-\nu} \gamma, \quad (81)$$

where  $E_r, \nu, \mu_r$  are the elastic modulus, Poisson's ratio and the shear modulus for the half-space.

Changing in Expression (80) to dimensionless variables:

$$\gamma = \frac{s}{l}; \quad l = \sqrt[3]{\frac{(1-\nu) D_n}{\mu_r}}; \quad n_0 = \frac{N_n l^3}{D_n a_0}; \quad \frac{r}{l} = \rho; \quad a_0 = l \sqrt{\frac{\rho_n}{D_n}}; \\ s_0 = p l a_0 \div \tau_0 = \frac{t}{l a_0}, \quad (82)$$

we will get the following expression for the transform of the deflection (80)

$$y_c^*(\rho, s_0) = \frac{Ql^2}{2\pi D_n} \int_0^\infty \frac{\xi J_0(\rho\xi) d\xi}{s_0^2 + n_0 s_0 + \xi(1 + \xi^2)}. \quad (83)$$

It is obvious that Integral (83) converges uniformly in  $s_0$  for  $\text{Re } s_0 > 0$ , since it is majorized for  $s_0 = 0$ . It is also obvious that the integrand increases monotonically as  $s_0 \rightarrow +0$ ,  $\text{Re } s_0 > 0$ , i.e., the conditions of the generalized Dini theorem are satisfied [19]. In conjunction with this, by writing the inverse transform of Expression (83) in terms of the Mellin transform, we can vary the order of integration, which corresponds to the feasibility of using ordinary transformation formulas [7, 9] for the integrand of Integral (83). In addition, it is possible to make limit transitions in the integrand. Taking the above into account, and using known rules of operational calculus [7, 9], we can write

$$\left. \begin{aligned} \lim_{\tau_0 \rightarrow \infty} y_c(\rho, \tau_0) &= \lim_{s_0 \rightarrow 0} y_c^*(\rho, s_0) = \frac{Ql^2}{2\pi D_n} \int_0^\infty \frac{J_0(\rho\xi) d\xi}{1 + \xi^2} = y_{c \text{ cr}}, \\ \lim_{\tau_0 \rightarrow 0} y_c(\rho, \tau_0) &= \lim_{s_0 \rightarrow \infty} y_c^*(\rho, s_0) = 0, \end{aligned} \right\} \quad (84)$$

and the inverse transform of the deflection then has the form

$$\left. \begin{aligned} y_c(\rho, \tau_0) &= y_{c \text{ cr}} - \frac{Ql^2}{2\pi D_n} e^{-\frac{n_0 \tau_0}{2}} \int_0^\infty f_1(\xi, \tau_0) \frac{J_0(\rho\xi) d\xi}{1 + \xi^2}; \\ f_1(\xi, \tau_0) &= \begin{cases} \frac{n_0}{2\omega} \text{sh } \omega\tau_0 + \text{ch } \omega\tau_0; & 0 \leq \xi < \xi_0; \\ \frac{n_0 \tau_0}{2} + 1; & \xi = \xi_0; \\ \frac{n_0}{2\omega} \sin \omega\tau_0 + \cos \omega\tau_0; & \xi_0 < \xi < \infty; \end{cases} \\ \omega = \omega(\xi) &= \sqrt{\left| \xi(1 + \xi^2) - \frac{n^2}{4} \right|}; \quad \xi_0^4 + \xi_0 - \frac{n^2}{4} = 0. \end{aligned} \right\} \quad (85a)$$

Replacing in Expressions (80) and (83) the force by the effect of the momentum, we will obtain similarly the deflection due to momentum  $S$ , i.e., the kernel of Problem (23):

$$\left. \begin{aligned} y_{c \text{ мпп}} &= \frac{Sl}{2\pi D_n a_0} e^{-\frac{n_0 \tau_0}{2}} \int_0^\infty \xi f_2(\xi, \tau_0) J_0(\rho\xi) d\xi; \\ f_2(\xi, \tau_0) &= \begin{cases} \frac{1}{\omega} \text{sh } \omega\tau_0; & 0 \leq \xi < \xi_0; \\ \tau_0; & \xi = \xi_0; \\ \frac{1}{\omega} \sin \omega\tau_0; & \xi_0 < \xi < \infty. \end{cases} \end{aligned} \right\} \quad (85b)$$

In order to evaluate them, the integrals in Expression (85) should be broken up into parts:

$$L = L_0 + L_1 + L_R; \quad L = \int_0^{\tau_0}; \quad L_0 = \int_0^{\tau_0}; \quad L_1 = \int_{\tau_0}^R; \quad L_R = \int_R^{\infty}, \quad (86)$$

where  $L_1$  contains the varying terms and for small  $\rho$  results in a form which makes possible the application of Filon formulas [15], while  $L_R$  is a correction. If, for  $\rho=0$ , the limit  $R$  is selected in a manner such that the integrands can be represented in asymptotic form, then integration of  $L_R$  by parts reduces to tabulated functions; for example, in the case of Eq. (85b):

$$L_R = -\frac{1}{2} \text{si}(x) + \frac{\tau_0}{2R} \left\{ \cos x - \sqrt{2\pi x} \left[ \frac{1}{2} - S(x) \right] \right\} - \left. \begin{aligned} & - \frac{\tau_0^2}{16R^3} [\sin x - x \text{ci}(x)] - \frac{\tau_0^3}{144R^3} \left\{ \cos x - 2x \sin x - \right. \\ & \left. - 2x\sqrt{2\pi x} \left[ \frac{1}{2} - C(x) \right] \right\} - \dots; \quad x = \tau_0 R^2, \end{aligned} \right\} \quad (87)$$

where  $\text{si}(x)$ ,  $\text{ci}(x)$  are the integral sine and cosine, while  $S(x)$ ,  $C(x)$  are Fresnel's integrals.

#### §4. Results of Numerical Computations

Computations using the above method were made on the M-20 EDC. Formula (15) and Algorithms (33) and (37) were programmed for the first collision. The calculations were made for an infinite (semi-infinite) beam on an elastic foundation for  $n=0$ ;  $k=0$  (Figs. 1-4), for an infinite (semi-infinite) beam with damping for  $\alpha=0$ ,  $k=0$  (Fig. 5) and for an infinite plate on a Winkler foundation for  $n=0$ ;  $k=0$  (Figs. 6-8). Figure 9 shows estimates of the effect of a constant force (weight) on the force of impact upon a plate for  $\beta=1.5$ ;  $b=0.125$ ;  $n=0$ , while Figs. 10 and 11 give the maxima of the impact force for beams and plates. Comparison of results using the Hertz-Timoshenko and Kil'chevskiy methods in the nonlinear case of  $\beta=1.5$  is given in Figs. 12 and 13 for beams and in Fig. 12 for concentrated masses. The correctness of results obtained in this work and the selection of pitch  $h$  for Algorithms (33) and (37) were estimated by comparing calculations on the basis of Formulas (15) and (33), (37) for  $\beta=1.0$ , when both solutions should be identical, as well as by comparing with Expression (40) for calculations for plates with  $\alpha=0$ . The exactness of summation in Expression (15) was  $1 \cdot 10^{-4}$ . The pitch  $h=0.1$ , assumed for calculations, has guaranteed an error of less than  $2 \cdot 10^{-3}$  in the impact force for beams and of  $1 \cdot 10^{-4}$  for plates.

It can be seen from results presented in Fig. 13 that the impact force calculated according to Kil'chevskiy is asymmetrical in the case of collision impact of two masses, which contradicts the statement of the fact (since we are considering a process without residual deformations) and the oscillograms presented for steel spheres in [4]. The experimental results published in [4] show that calculations according to Hertz are in agreement with the experimentally obtained impact force, while calculations according to Kil'chevskiy give a higher impact force than according to Hertz-Timoshenko: +9.2% for mass (Fig. 13) and +10.6% for beams

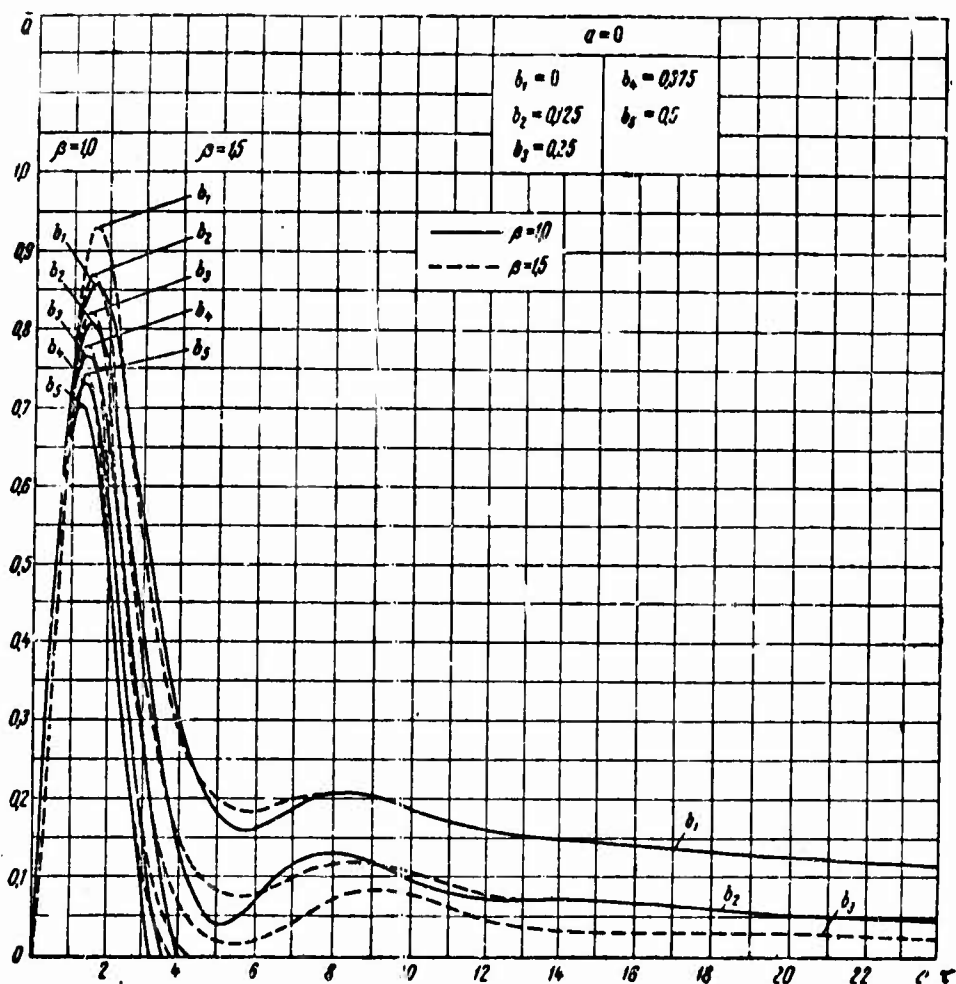


Fig. 1

(Fig. 12). The period of collision impact of two masses according to Kil'chevskiy for  $\beta=1.5$  is determined from Formula (52). Since in [3] first 2 and then 3 terms were retained in the series for  $Q(\tau)$ , then the root  $\bar{Q}(\tau)=0$  was calculated by us also for different  $j$  (by the inverse interpolation method with  $\Delta\tau=0.1$ ). Roots  $\bar{Q}(\tau)=0$  are presented in Fig. 13. It can be seen from this that the value of root  $\bar{Q}(\tau)$ , calculated in [3] and equal to 2.977, is incorrect and is a result of an unsuccessful iterative process used in [3]. It follows from Fig. 13 that the interval of the impact calculated according to Hertz and Kil'chevskiy is practically identical (3.203 and 3.218), i.e., the interpretation of experimental results in [20], in which the experimental impact interval was found smaller than that calculated according to Hertz, is given in [20] on the basis of the incorrect result of [3]. It should also be noted that due to the great amount of machine time required for  $a, b \neq 0$  Expansion (15) can be used only for calculating the first maximum of the impact force.

It follows from the results of calculating the first collision for infinite beams presented in Figs. 1-5 and 10 that:

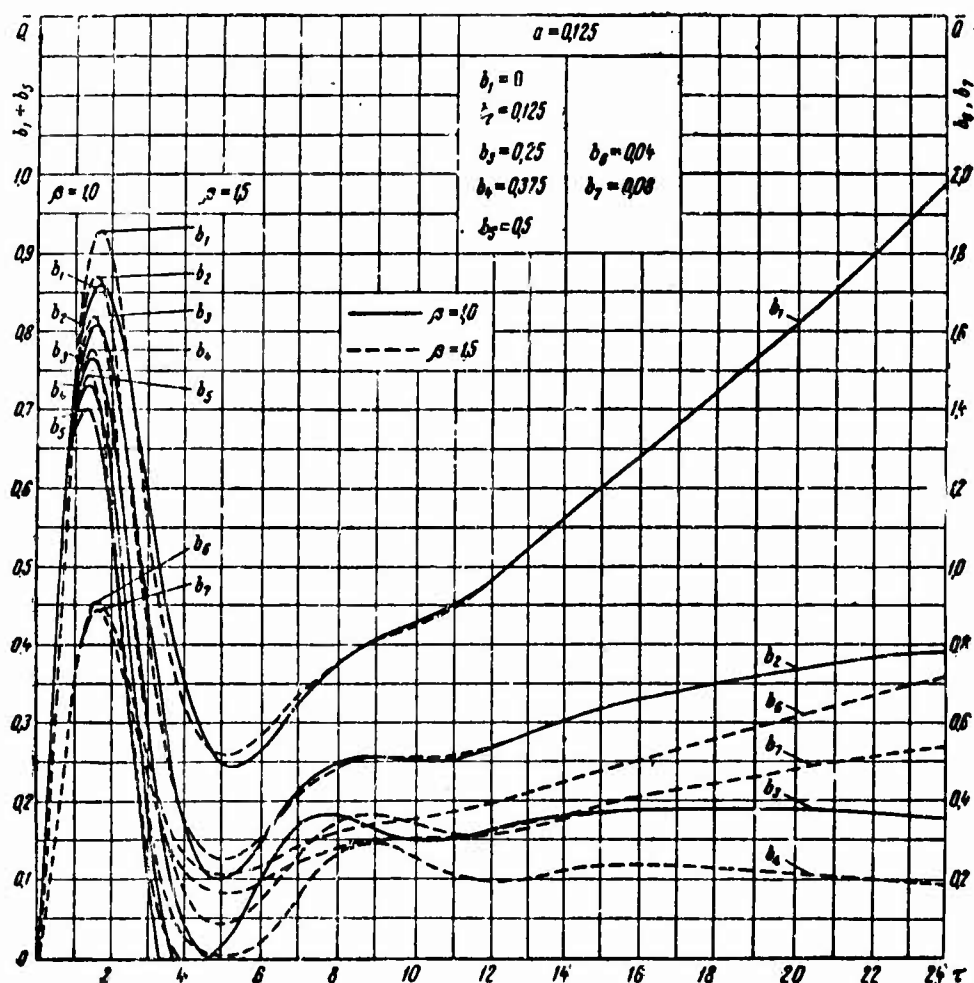


Fig. 2

1) The first maximum of the impact force corresponds to impact upon a beam without a foundation ( $a=0$ ) and, consequently, it can be interpreted as an impact of mass  $M$  and the reduced mass of the infinite beam, separated by a contact spring. For the first maximum of the impact force in the assumed system of dimensionless quantities one discovers that the linear ( $\beta=1.0$ ) and nonlinear ( $\beta=1.5$ ) (Fig. 10) differ by about 8%. The first maximum of the impact force depends very little on the relative stiffness of the beam on impact for  $a < 0.375$  and varies somewhat more strongly with variations in the damping of the beam. The first maximum characterizes the period of "self-impact" for passive systems, when the result is not as yet affected by the vibrations of this system.

2) The second maximum of the impact force is related to the motion of the passive system (beam) and mass  $M$  toward one another and hence depends on the period of vibrations of the system (coefficient  $a$ ) and loss of velocity of impacting mass  $M$  due to the momentum of the force (coefficient  $b$ ); here for  $b=0$  ( $M = \infty$ ) the maximum increases beyond bounds, since then this loss of velocity does not take place. Consequently, a case is possible for small  $b$  (large  $M$ ) when the second maximum of the impact force in the first collision will exceed the first maximum.

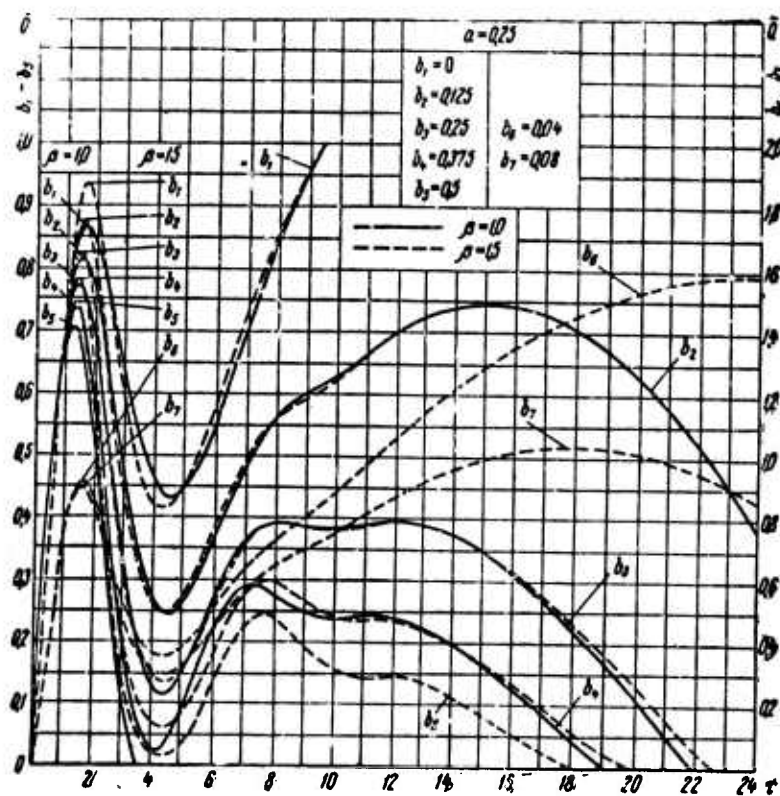


Fig. 3

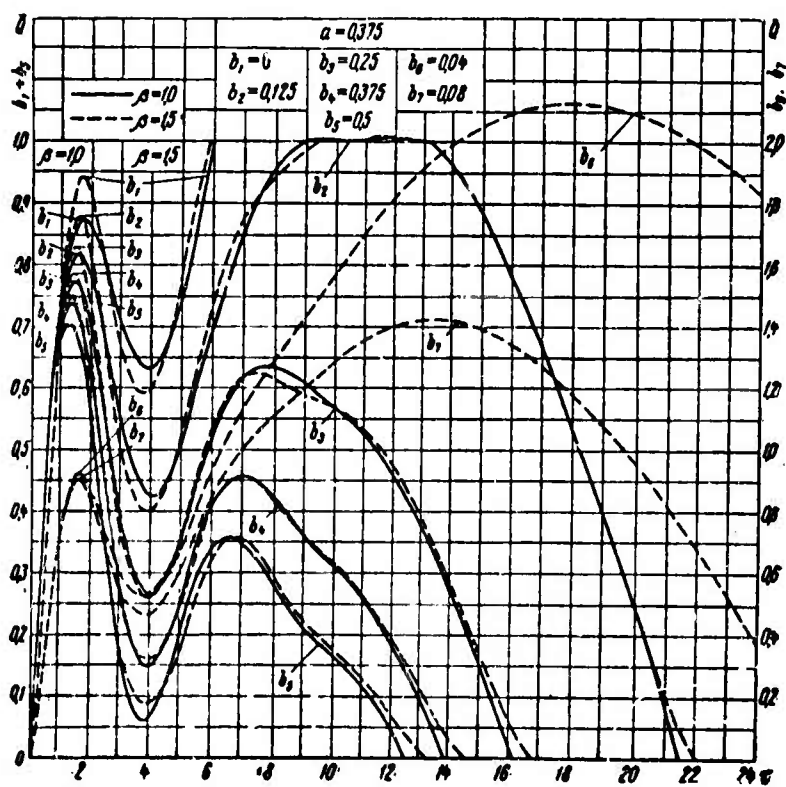


Fig. 4

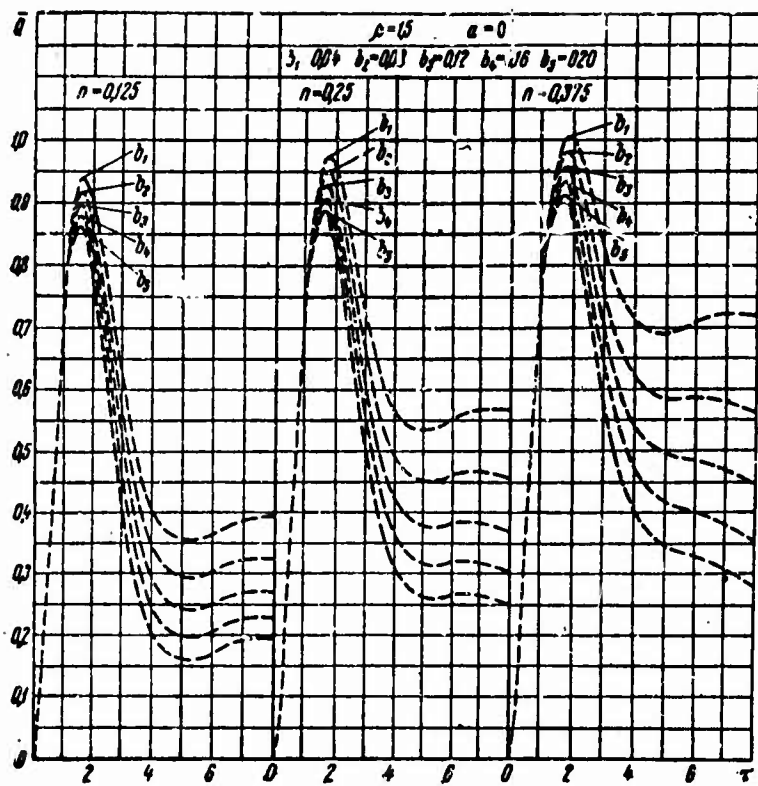


Fig. 5

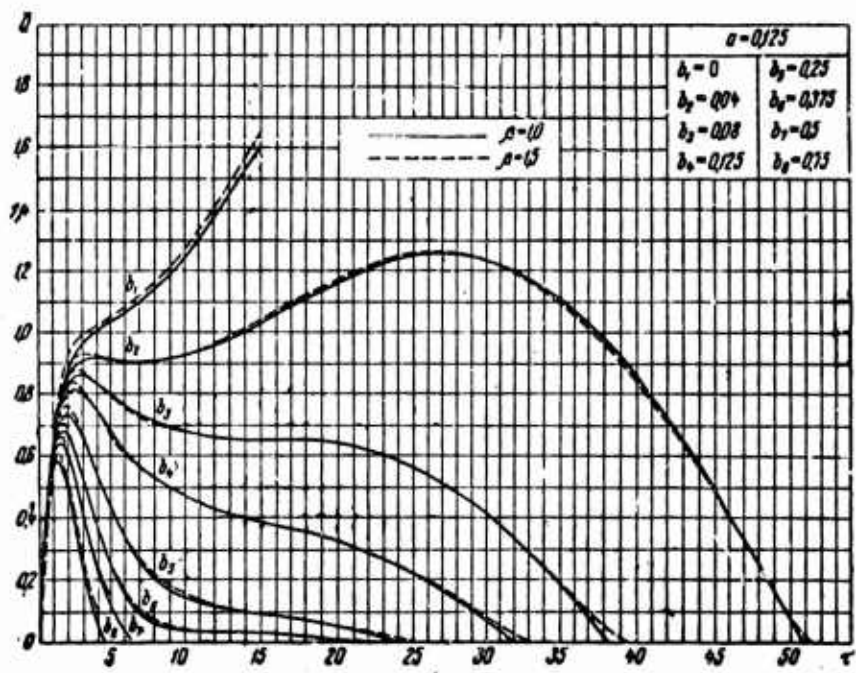


Fig. 6

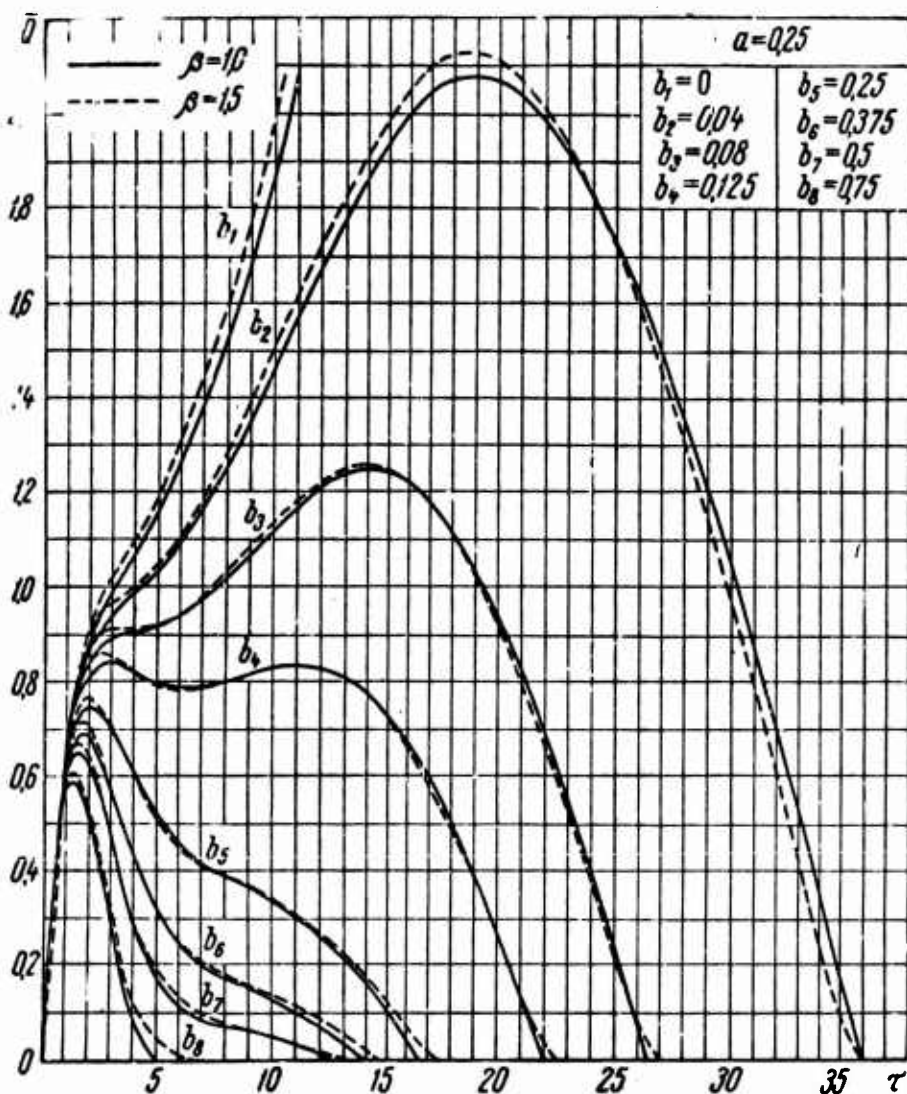


Fig. 7

We note that the second maximum of the dimensionless impact force, which is determined by induced vibrations of the passive system, turns out to be practically identical for infinite beams in the linear ( $\beta=1.0$ ) and nonlinear ( $\beta=1.5$ ) cases.

This circumstance is also verified in calculating impact upon infinite plates, with the greatest error from replacing the nonlinear case ( $\beta=1.5$ ) by the linear ( $\beta=1.0$ ) for the second maximum of the impact force obtained for plates (Fig. 11 -  $a=0.375$ ;  $b=0.04$ ) and is equal to 4.7%. In the remaining cases this error is less than 2%. For infinite plates the previously made statement on the relation between the value of the first and second maximum of the impact force for small  $b$  is verified (Figs. 6-8). The first maximum of the impact force for infinite plates is expressed much more weakly than for beams, and the effect on it of index  $\beta$  and coefficient  $a$  is practically unsubstantial (Fig. 10). Comparison of impact for concentrated masses, beams and plates shows that the intensity of the first maximum of the impact force depends on increases in the

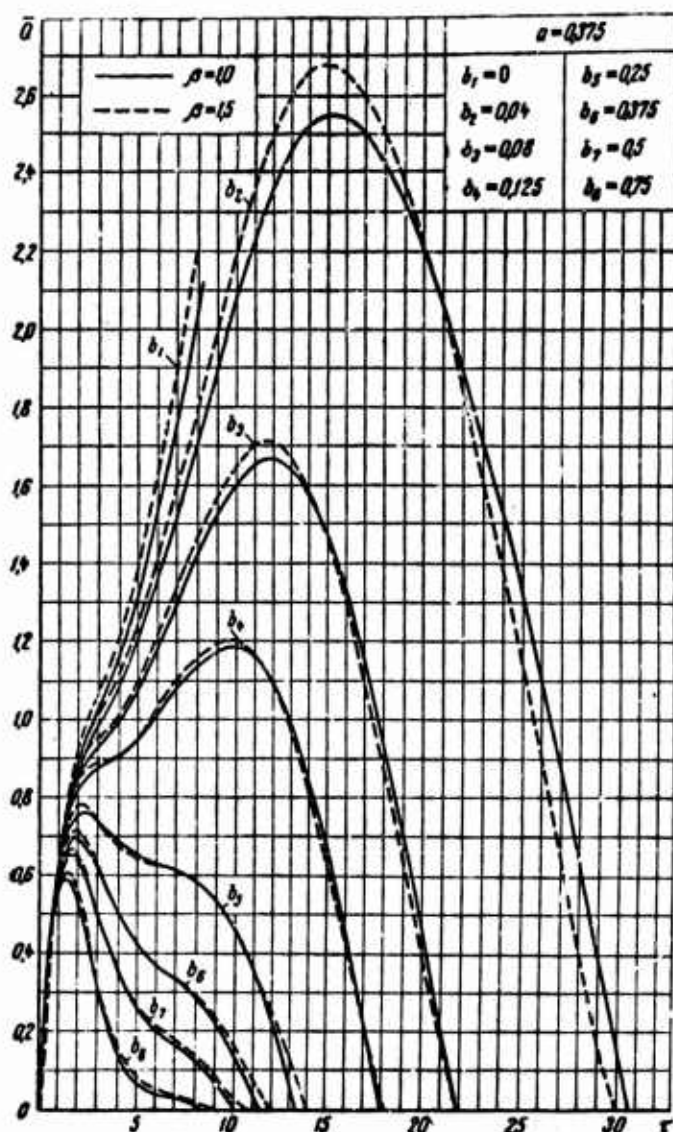


Fig. 8

deflections of the passive system during the initial period of the impact, i.e., on indicator  $\alpha$  in Expansion (5). For this reason, the first maximum is more pronounced on impact of concentrated masses ( $\alpha=2$ ) and is least pronounced for impact upon plates ( $\alpha=1$ ). We note that the effect of the constant force (weight) on the impact force (coefficient  $k$ ) can be substantial, particularly for the second maximum of the impact force (Fig. 9).

It can be seen from the above calculations for the first collision that the impact force is characterized by not less than 5 parameters ( $a, b, n, k$ , and  $\beta$ ); here for small  $b$  it is the second maximum of the impact force which is more important from the point of view of contact strength, with the first maximum being more important for large  $b$ . It is of substance in analyzing the above factors that the second maximum of the dimensionless force in beams and plates is practically independent of the nonlinearity index  $\beta$ . The following two methods of approximate calculation, re-

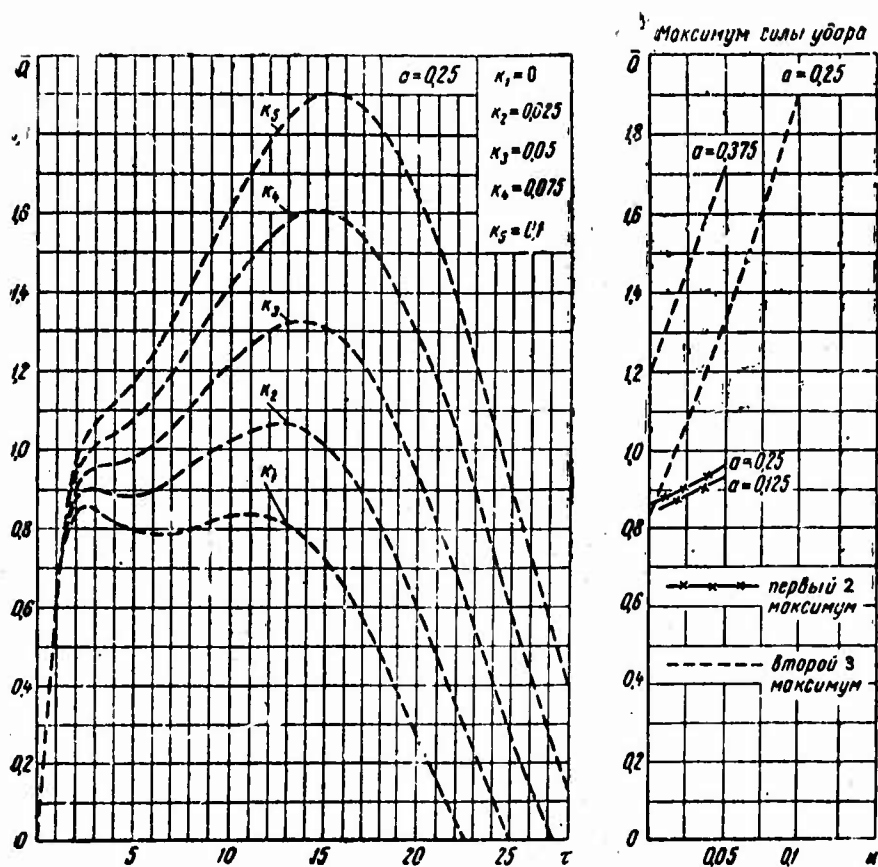


Fig. 9. 1) Maximum of impact force; 2) first maximum; 3) second maximum.

lated to the fact that the first maximum arises at the "self-impact" stage, may be useful in calculating the first maximum.

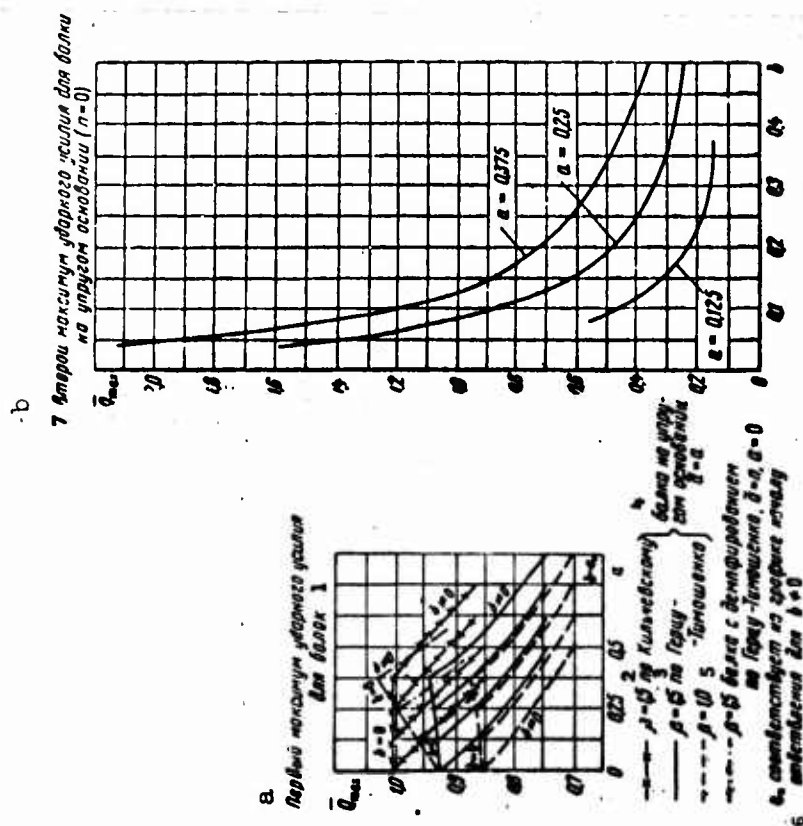


Fig. 10. 1) First maximum of impact force for beams; 2) according to Kil'chevskiy; 3) according to Hertz-Timoshenko; 4) beam on elastic foundation; 5) beam with damping according to Hertz-Timoshenko; 6)  $a_1$  corresponds on the graph to the start of branching off for  $b \neq 0$ ; 7) second maximum of impact force for a beam on elastic foundation.

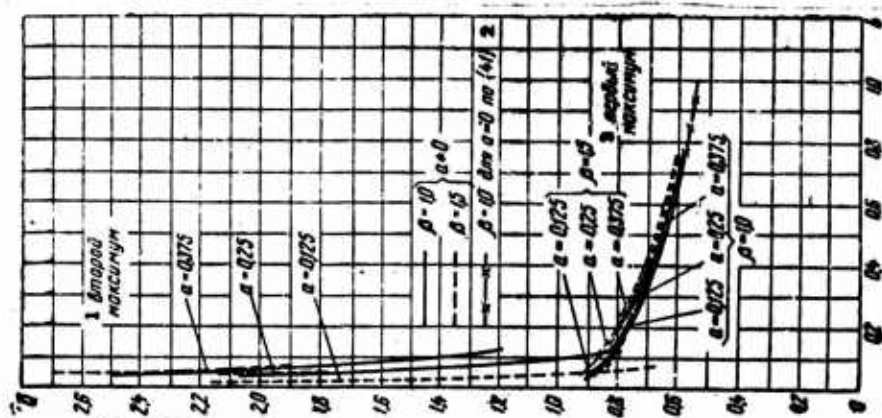


Fig. 11. 1) Second maximum; 2) for  $a = 0$  from (41); 3) first maximum.

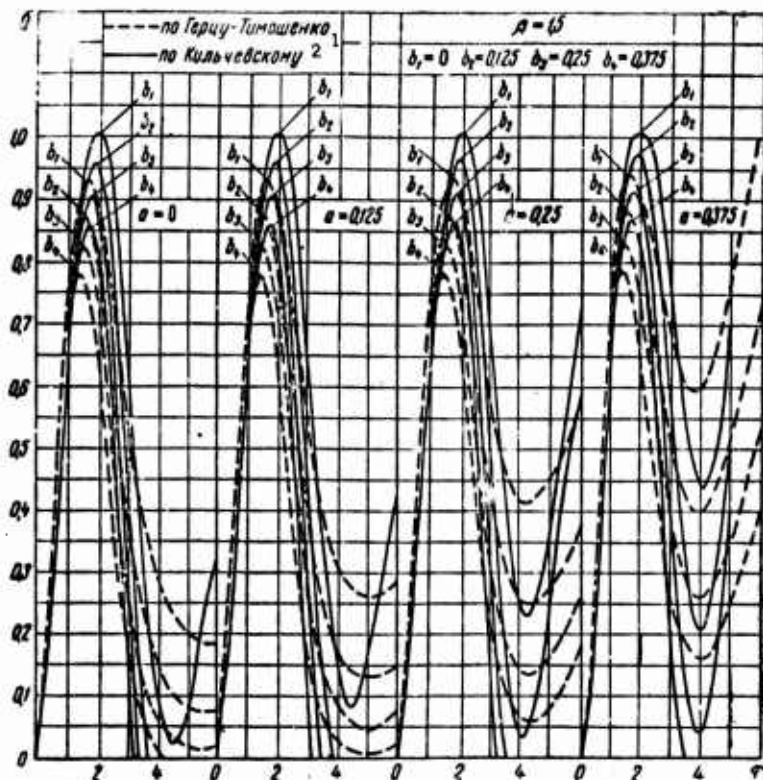


Fig. 12. 1) According to Hertz-Timoshenko; 2) according to Kil'chevskiy.

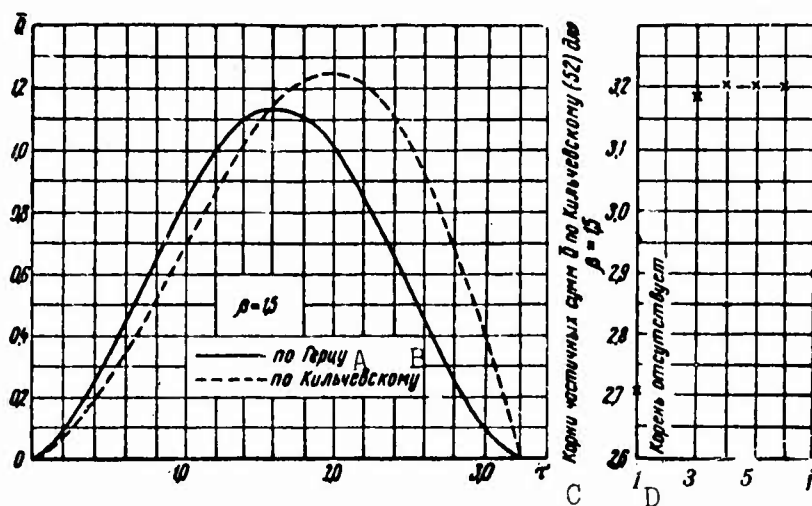


Fig. 13. A) According to Hertz; B) according to Kil'chevskiy; C) roots of partial sums  $\bar{Q}$  according to Kil'chevskiy (52) for  $\beta = 1.5$ ; D) there is no root.

#### a. Approximate consideration of the effect of nonlinearity index $\beta$

It was suggested in [1], for simplifying the nonlinear problem, to solve the linear problem with a contact stiffness selected from assuming an equal energy of local deformation in both cases for a maximum contact force.

The work of the contact force in the linear case ( $\beta = 1$ ), i.e.,  $A_1$  and in the nonlinear case ( $\beta \neq 1$ ), i.e.,  $A_\beta$  will, using Relationship (1), be written in the form

$$A_1 = \frac{1}{c_e} \int_0^{Q_{\max}} Q dQ = \frac{Q_{\max}^2}{2c_e}; \quad A_\beta = \frac{1}{\beta c_e^{1/\beta}} \int_0^{Q_{\max}} Q^{1/\beta} dQ = \frac{Q_{\max}^{1/\beta+1}}{c_e^{1/\beta} (1+\beta)}, \quad (88)$$

where  $c_e$  is the equivalent linear stiffness.

It is easy to establish from geometric considerations that when  $A_1 = A_\beta$  and for identical  $Q_{\max}$  the maximum contact deformations cannot be identical, while when these deformations are equal, the values of  $Q_{\max}$  are different. Consequently, when  $A_1 = A_\beta$  the time at which the maximum impact force arises may be distorted, which can be seen, in particular, from the example in [1].

The use of iteration is suggested in [1] for calculating the impact force in the nonlinear case. We shall proceed differently, by assuming the solution of the linear problem  $Q_{1\max}$  to be known; then, according to (15),

$$Q_{2\max} = c_e v a_1; \quad Q_{1\max} = Q_{2\max} (c_1 = c_e). \quad (89)$$

The use of Expression (89) and the condition that  $A_1 = A_\beta$  yields for the equivalent contact stiffness for identical  $Q_{\max}$

$$c_{e1} = \bar{c}_{e1}^{\alpha+\beta-1} \sqrt[\alpha]{c_k^\alpha c^{\beta-1} v^{\alpha(\beta-1)}}; \quad \bar{c}_{e1} = \left[ \left( \frac{1+\beta}{2} \right)^\beta \bar{Q}_{1\max}^{\beta-1} \right]^{\frac{\alpha}{\alpha+\beta-1}}. \quad (90)$$

Substitution of (90) into (89) yields the sought maximum of the impact force for  $\beta \neq 1$

$$Q_{\max} = E_Q \bar{Q}_{1\max}; \quad \bar{Q}_{1\max} = \left[ \left( \frac{1+\beta}{2} \right)^{\alpha-1} \bar{Q}_{1\max}^\alpha \right]^{\frac{\beta}{\alpha+\beta-1}}. \quad (91)$$

If, however, it is assumed that  $y_{k\max}$  is the same for  $\beta = 1$  and  $\beta \neq 1$ , then we get for the equivalent contact stiffness  $c_e$

$$c_{e2} = \bar{c}_{e2}^{\alpha+\beta-1} \sqrt[\alpha]{c_k^\alpha c^{\beta-1} v^{\alpha(\beta-1)}}; \quad \bar{c}_{e2} = \left[ \frac{2}{1+\beta} \bar{Q}_{1\max}^{\beta-1} \right]^{\frac{\alpha}{\alpha+\beta-1}}. \quad (92)$$

Substitution of  $c_e$  into the formula for the time of impact  $t = a_1 \tau$  (10) yields the following expression for recalculating the time at which maximum impact force occurs for  $\beta \neq 1$ :

$$t_{\max} = a_1 \tau_{1\max}; \quad \tau_{1\max1} = \bar{c}_{e1}^{-\frac{1}{\alpha}} \tau_{1\max}; \quad \tau_{1\max2} = \bar{c}_{e2}^{-\frac{1}{\alpha}} \tau_{1\max}. \quad (93)$$

We consider recalculation using Formulas (91) and (93) with passive systems with different  $\alpha$ . Using Expression (53) for linear collision impact of two masses, we get from Formulas (91) and (93)

$$\alpha = 2; \bar{Q}_{\max} = \left( \frac{1+\beta}{2} \right)^{\frac{\beta}{1+\beta}}; \tau_{\max 1} = \frac{\pi}{2} \left( \frac{2}{1+\beta} \right)^{\frac{\beta}{1+\beta}};$$

$$\tau_{\max 2} = \frac{\pi}{2} \left( \frac{1+\beta}{2} \right)^{\frac{1}{1+\beta}}; \quad (94)$$

the  $\bar{Q}_{\max}$  is identical with (54), the exact Hertz solution, while  $\tau_{\max}$  gives for  $\beta=1.5$  errors of -14.6% ( $\tau_{\max 1}$ ) and +6.7% ( $\tau_{\max 2}$ ).

Results of recalculations using Formulas (91) and (93) for beams ( $\alpha = \frac{3}{2}$ ;  $a=0.375$ ) are presented in the table from which it follows that Expression (91) is in good agreement with calculations using the EDC, while Expression (93) introduces an error. For plates ( $\alpha=1$ ) it follows from Formula (91) that  $\bar{Q}_{\max} = Q_{\pi \max}$ , i.e., when  $\alpha=1$  the energy equality  $A_{\pi}=A$  is satisfied for any  $\beta$ . Above we have noted that the first maximum of the impact force for plates depends little on index  $\beta$ , with replacement of  $\bar{Q}_{\max}$  ( $\beta=1.5$ ) by  $Q_{\pi \max}$  according to (91) giving an error of -3% (Fig. 11).

It is seen from the above results that recalculation using Formula (91) is in good agreement for the value of the first maximum of the impact force; however, the  $\tau$  curve must be distorted. It can be seen from the data presented in tables and figures that to approximately obtain the entire curve of the impact force for  $\beta \neq 1$  it is necessary to calculate the dimensionless impact force for  $\beta=1.0$  and then to introduce a correction only into the first maximum of (91).

*b. Approximate consideration of the effect of mass M (coefficient b)*

Since, as was noted above, the first maximum of the impact force for infinite beams ( $\alpha = \frac{3}{2}$ ) can be interpreted as the collision impact of two masses, we now determine  $M_e$ , the reduced mass of the beam, for the first maximum of the impact force by setting equal the maxima of the impact force  $Q_{b_0}$  for the beam for coefficient  $b_0$  (impacting mass  $M_{b_0}$ ) and impact force  $Q_M$  for collision impact of two masses according to Hertz. From Expression (15) for  $\alpha = \frac{3}{2}$  and from Formula (54) it follows

$$c_k (va_1)^{\beta} \bar{Q}_{b_0 \max} = c_k (va_{1M})^{\beta} \left( \frac{1+\beta}{2} \right)^{\frac{\beta}{1+\beta}};$$

$$a_1 = \sqrt[\beta + \frac{1}{2}]{c_k^{-1} c v^{1-\beta}}; a_{1M} = \sqrt[\beta + 1]{c_k^{-1} M_e v^{1-\beta}}; M_e = \frac{M_{b_0} M_{b_0}}{M_{b_0} + M_{b_0}}. \quad (95)$$

Introducing, according to (11), the dimensionless mass  $\bar{M}$

$$M = ca_1^{\beta} \bar{M} \quad (96)$$

and using Formulas (11), (95) and (96), we will get the dimensionless reduced mass of the beam  $\bar{M}_e$  in the form

$$\bar{M}_b = \frac{\bar{M}_z}{1 - b_0 \bar{M}_z}; \quad \bar{M}_z = \frac{\bar{M}_z \bar{M}_{b_0}}{\bar{M}_z + \bar{M}_{b_0}} = \frac{2}{1 + \beta} \bar{Q}_{b_0 \max}^{\frac{1+\beta}{\beta}}; \quad \bar{M}_{b_0} = \frac{1}{b_0}; \quad (97)$$

If we assume that the value of  $\bar{M}_e$  remains unchanged within some range of variation of coefficient  $b$ , then substitution of  $\bar{M}_b$  and  $\bar{M}_e$  according to Expressions (96) and (97) into the Hertz formula for impact collision of two masses yields the following expression for the maximum impact force for coefficient  $b$  in terms of  $\bar{Q}_{b_0 \max}$ :

$$Q_{b \max} = E_Q \bar{Q}_{b \max}; \quad \bar{Q}_{b \max} = \frac{\bar{Q}_{b_0 \max}}{\left[1 + \frac{2}{1 + \beta} (b - b_0) \bar{Q}_{b_0 \max}^{\frac{1+\beta}{\beta}}\right]^{\frac{\beta}{1+\beta}}}. \quad (98)$$

Results of recalculation according to Formula (98) are presented in the table and are in good agreement with EDC calculations for a pitch of  $b$  equal to 0.5 (error of up to 5%).

In the linear case ( $\beta=1$ ) it is possible to give, in addition to Formula (98), still another formula for calculation according to coefficient  $b$ . According to Expression (16), the transform of the impact force with coefficients  $b$  and  $b_0$  has the form

$$\bar{Q}_b^* = \frac{1}{s \left[1 + \frac{b_0}{s^2} + \bar{\psi}_1^*\right]}; \quad \bar{Q}_b^* = \frac{1}{s \left[1 + \frac{b}{s^2} + \bar{\psi}_1^*\right]} = \frac{\bar{Q}_{b_0}^*}{1 + \frac{b - b_0}{s} \bar{Q}_{b_0}^*}. \quad (99)$$

Assuming  $\bar{Q}_{b_0} = A_0 \sin \omega_0 \tau$ ;  $\bar{Q}_{b_0}^* = A_0 \frac{s \omega_0}{s^2 + \omega_0^2}$ , we get

$$\bar{Q}_b = A \sin \omega \tau; \quad A = \frac{A_0}{\Delta}; \quad \omega = \omega_0 \Delta; \quad \Delta = \sqrt{1 + (b - b_0) \frac{A_0}{\omega_0}}. \quad (100)$$

It is entirely obvious that the method of approximate consideration of the coefficient  $b$ , suggested above, is unsuitable for infinite plates ( $\alpha=1$ ), since the first maximum of the impact force in the case of  $\alpha=1$  is interpreted as impact in the system of a hydraulic shock absorber (see the problem on page 44). Hence, for approximate consideration of the effect of coefficient  $b$ , one should in this case use Formula (41) which corresponds to impact upon a plate for  $\alpha=0$  and  $\beta=1.0$ , as well as the previously noted little-felt effect of nonlinearity indicator  $\beta$  on the first maximum of the impact force for plates. The curve of the maximum impact force calculated according to Formula (41) is presented in Fig. 11. The use of this curve instead of  $Q_{\max}$ , calculated on an EDC for the nonlinear case of  $\beta=1.5$  yields for  $a \leq 0.375$  the greatest error of -6% for  $b=0.25$ ; here, with an increase in  $b$ , this error becomes smaller (to -3% for  $b=0.75$ ).

It follows from what was said in Subsections 4"a" and "b" that the number of versions to be calculated in estimating the impact contact strength for the first collision can be substantially reduced for the dimensionless variables used in this work in the

case of passive systems with  $\alpha=1; 3/2; 2$ . For infinite beams and plates use can be made of data on the maximum dimensionless impact force, presented in Figs. 10-11; here, transition to the dimensional impact force takes place in accordance with notation of Formula (15). Certain considerations were put forward on page 42 relative to the intensity of recurrent impacts; however, to estimate the strength on recurrent impacts, it is necessary to determine  $n$ , the damping coefficient of the passive system, for which for a given  $\alpha$  the recurrent impacts are found to be weaker than the first impact. Here it is necessary to take into account the effect of the constant force (the force of the weight, i.e., coefficient  $k$ ). As follows from problem 3"с," the damping source for beams and plates on an elastic foundation is dissipation of energy as a result of the wave process in the foundation.

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#### Transliterated Symbols

- |    |  |
|----|--|
| 41 | $\varepsilon = e = \text{ekvivalentnyy} = \text{equivalent}$ |
| 47 | $b = b = \text{balka} = \text{beam}$                         |
| 48 | $d = d = \text{dinamicheskiiy} = \text{dynamic}$             |
| 48 | $изг = izg = \text{izgib} = \text{flexure}$                  |

54       ст = st = staticheskiy = static  
63       н = n = nachal'nyy = initial  
65       л = l = lineynyy = linear

## ON THE INFLUENCE OF WEIGHT-SUPPORTING SPRINGS ON THE DYNAMIC EFFECT OF A MOVING LOAD

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It is usually assumed in calculating the dynamic effect of a moving load with consideration of its mass that the action of this load is directly transferred to the structure [1]. Actually, however, the load of railroad rolling stock or motor vehicles usually consists of two parts, namely, the underspring part, consisting of the car and locomotive undercarriages, and the abovespring part, consisting of the bodies and the payload.

The pressure from the underspring part is transmitted to the structure directly, while the pressure of the abovespring part is transmitted through load-supporting springs, which soften the shocks and somewhat modify the character of the dynamic effect of the moving load. The present article is concerned precisely with this effect of load-supporting springs on the dynamic effect of a moving load, moving along a smooth track.

Thus, let a load, with underspring mass and weight of  $M_1$  and  $P_1$  and an abovespring mass and weight of  $M_2$  and  $P_2$ , respectively, be moving along a smooth, simply supported beam.

At the instant the load is put onto the beam, the static deformation of the spring is  $\Delta_{st}$ , the deflection of the beam and its velocity are zero. The stiffness factor of the spring, i.e., the reciprocal of the static deformation produced in the spring by unit force, will be denoted by  $k$ . The beam's deflection in any section is denoted by  $u(x, t)$  and, restricting ourselves to the first harmonic, we assume

$$u(x, t) = f(t) \sin \frac{\pi x}{l}. \quad (1)$$

The displacement of the center of gravity of the abovespring load relative to its position on the undeformed spring is denoted by  $w(t)$ . By virtue of the above,  $w(0) = \Delta_{st}$ . A schematic of the loading of a beam by a load, partially supported by a load-supporting spring in the form of a simple spring, is shown in Figs. 1 and 2.

The beam under consideration has, in general, an infinite number of degrees of freedom; however, by specifying the shape of the elastic curve by Formula (1), we thus reduce the number of

degrees of freedom of the system consisting of the beam and of underspring mass  $M_1$  to one.

The abovespring part of the load, whose mass is  $M_2$ , also has one degree of freedom, since it is free to perform vibrations, independent of the beam, along the axis of the spring. In the statement under consideration, the system being considered here thus has two degrees of freedom.

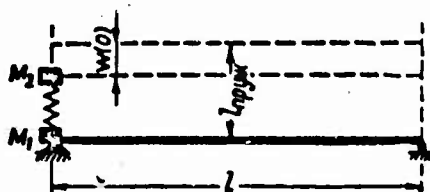


Fig. 1



Fig. 2

The motion of this system can be described by two independent parameters  $f(t)$  and  $w(t)$ , the first of which characterizes the time-variable deflection of the beam's midspan, while the other describes the displacement of mass  $M_2$  along the axis of the spring.

We first set up the equation of motion of abovespring mass  $M_2$ . This latter mass is acted upon by three forces, namely: the weight force  $P_2$ , inertia force  $M_2 \frac{d^2w}{dt^2}$  and spring reaction  $-k[w(t) - u(vt, t)]$ .

Using the d'Alembert principle, we can write

$$\Sigma Y = -M_2 \frac{d^2w}{dt^2} + P_2 - k(w - u) = 0.$$

In the given case,  $u$  is the ordinate of the elastic curve of the beam beneath the load and, consequently, using Formula (1), we can write the equation of motion of mass  $M_2$  in the form

$$M_2 \frac{d^2w(t)}{dt^2} + kw(t) - kf \sin \frac{\pi vt}{e} = P_2. \quad (2)$$

The second equation characterizes the motion of the beam together with underspring mass  $M_1$ . To set up this equation we use the differential equation of equilibrium for a beam element

$$EJ \frac{\partial^4 u(x, t)}{\partial x^4} - q(x, t) = 0. \quad (3)$$

We assume that the moving load is distributed over some, quite small, segment with length  $\epsilon$ . Then, disregarding the weight of the beam proper, everywhere, except for this segment, we can write

$$q(x, t) = -m \frac{\partial^2 u(x, t)}{\partial t^2}. \quad (4)$$

On the other hand, inside this segment we have

$$q(x, t) = -m \frac{\partial^2 u(x, t)}{\partial t^2} - \frac{M}{\varepsilon} \cdot \frac{d^2 u(x, t)}{dt^2} + \frac{k}{\varepsilon} (w - u) + \frac{P_1}{\varepsilon},$$

where

$$\frac{d^2 u(vt, t)}{dt^2} = \frac{\partial^2 u}{\partial t^2} + 2v \frac{\partial^2 u}{\partial x \partial t} + v^2 \frac{\partial^2 u}{\partial x^2}. \quad (5)$$

We solve Eq. (3) by the Bubnov-Galerkin method; then, taking into account (1), we can write

$$\int_0^l \left[ EJ \frac{\partial^4 u(x, t)}{\partial x^4} - q(x, t) \right] \sin \frac{\pi x}{l} dx = 0.$$

Taking into account Relationships (1), (4) and (5) and integrating after transformations, we get the following differential equation of motion of the beam and mass  $M_2$ :

$$\begin{aligned} & (1 + 2\mu_1 \sin^2 \pi \xi) f''(t) + 2\mu_1 \sin 2\pi \xi f'(\xi) + \\ & + (\kappa - 2\mu_1 \pi^2 \sin^2 \pi \xi) f(\xi) - \delta \sin \pi \xi w(\xi) + \\ & + \delta \sin^2 \pi \xi f(\xi) = \frac{2P_1 l}{mv^2} \sin \pi \xi, \end{aligned} \quad (6)$$

where

$$\begin{aligned} \mu_1 &= \frac{M_1}{ml}; \quad \mu_2 = \frac{M_2}{ml}; \quad \kappa = \frac{\omega_1^2 l^2}{v^2}; \\ \omega_1 &= \frac{\pi^2}{l^2} \sqrt{\frac{EJ}{m}}; \quad \beta = \frac{f_{\text{б.а.к.н.}}^{\text{с.т.}}}{\Delta_{\text{с.т.}}}; \\ \delta &= \frac{2k}{ml} \cdot \frac{l^2}{v^2} = \kappa \beta \end{aligned} \quad (7)$$

and

$$\xi = \frac{vt}{l}. \quad (8)$$

As was pointed out above, the purpose of the present study is the problem of the influence of load-supporting springs on the magnitude of the dynamic effect of a moving load.

For this purpose, it is pertinent to compare the magnification factor for load-supporting springs of different stiffness attendant to the motion of one, fully spring-supported load (i.e., assuming  $M_1 = 0$ ) with the magnification factor for a load fully nonspring-supported.

System of equations (2) and (6) for  $M_1 = 0$  simplifies and can be written as

$$\omega_0'(\xi) + \lambda \omega_0(\xi) - \lambda \sin \pi \xi f_0(\xi) = a; \quad (9a)$$

$$f_0'(\xi) + (\kappa + \delta \sin^2 \pi \xi) f_0(\xi) - \delta \sin \pi \xi \omega_0(\xi) = 0. \quad (9b)$$

A transition was made in System (7) to dimensionless variables  $w_0$  and  $f_0$  by means of the substitutions

where

$$w(\xi) = f_{cr} w_0(\xi) \text{ and } f(\xi) = f_{cr} f_0(\xi),$$

$$f_{cr} = \frac{2Pl^3}{\pi^4 EJ} = Pf_{cr}^0 \approx \frac{Pl^3}{48 EJ}, \quad (10)$$

and also the following designations were introduced:

$$\left. \begin{aligned} \lambda = \frac{k}{M_1} \cdot \frac{l^3}{v^3} &= \frac{1}{2} \delta \frac{1}{\mu_1}; \quad \alpha = \frac{v}{v_{кр}}; \quad v_{кр} = \frac{\pi}{l} \sqrt{\frac{EJ}{m}} \cdot \frac{1}{\sqrt{1+\mu_2}}; \\ x_0 &= \frac{\pi^3}{\alpha^3} (1+2\mu_2); \quad a = \frac{P_2 l^3}{f_{cr}^0 M_2 v^3} = \frac{\pi^3}{2\mu_2 \alpha^3} (1+2\mu_2). \end{aligned} \right\} \quad (11)$$

System of equations (9) should be integrated with the initial conditions

$$f_0(0) = f_0'(0) = w_0'(0) = 0; \quad w_0(0) = \frac{1}{\beta}. \quad (12)$$

The last of Eqs. (12) expresses the requirement that at the start of motion the deformation of the spring be equal to its static deformation produced by compressive force  $P_2$ .

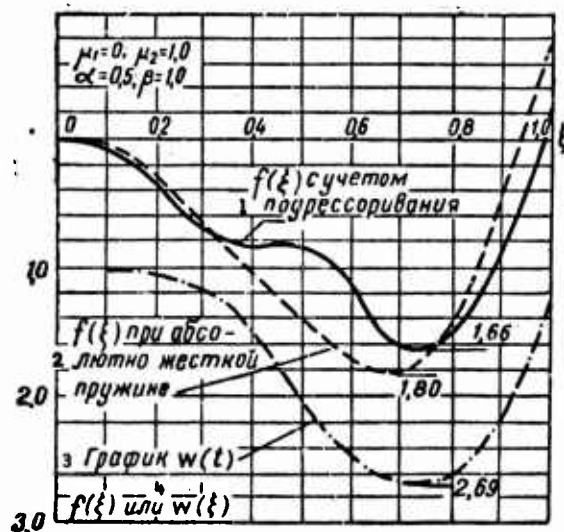


Fig. 3. 1) Taking spring support into account; 2) for a perfectly stiff spring; 3) graph of; 4) or.

System of equations (9) with initial conditions (12) can be solved by numerical methods. In this case both Eqs. (9) should be integrated simultaneously, extending the tables by means of Störmer's formula

$$\Delta^2 f_{n-1} = \eta_n + \frac{1}{12} \left[ \Delta^2 \eta_{n-2} + \Delta^2 \eta_{n-3} + \frac{19}{20} \Delta^4 \eta_{n-4} \right].$$

For Eq. (9a) the term  $\eta_i$  is meant to denote the expression

$$\eta_i^{(w)} = h^2 [a - \lambda w_0(\xi_i) + \lambda \sin \pi \xi_i f_0(\xi_i)],$$

and for Eq. (9b) it denotes the expression

$$\eta_i^{(w)} = h^2 [\delta \sin \pi \xi_i v_0(\xi_i) - (\kappa + \delta \sin^2 \pi \xi_i) f_0(\xi_i)],$$

where  $h$  is the interval of integration.

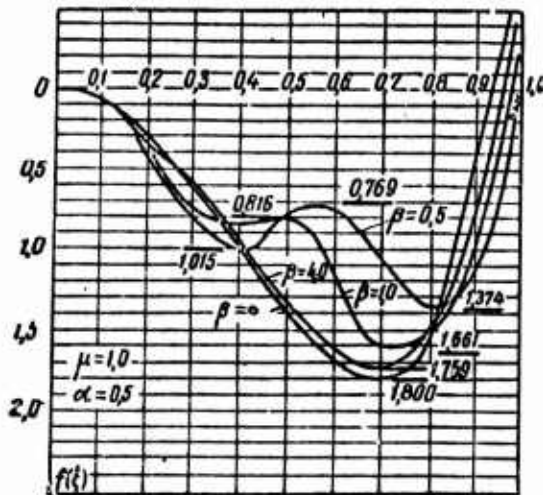


Fig. 4

The initial rows of the table can be filled either by successive approximations or by means of series, which are power expansions of functions  $f_0(\xi)$  and  $w_0(\xi)$  in the vicinity of the zero point. These series are obtained most simply by the method of indeterminate coefficients

$$\begin{aligned} w_0(\xi) &= D_0 - D_6 \xi^6; \\ f_0(\xi) &= C_3 \xi^3 - C_6 \xi^6. \end{aligned}$$

where

$$\begin{aligned} D_0 &= \beta; \quad D_6 = -\frac{1}{180} \pi^2 \delta \beta; \\ C_3 &= \frac{\delta \pi}{6} \beta; \quad C_6 = -\frac{\delta \pi}{120} (\kappa + \pi^2) \beta. \end{aligned}$$

Integrating Eq. (9) numerically, we will find for each position of the load the deflection  $f_0(\xi)$  and displacement  $w_0(\xi)$  for load  $M_2$ . Knowing them it is not too difficult to determine the total depression of the spring

$$\delta(\xi) = f_{cr} [w_0(\xi) - f_0(\xi) \sin \pi \xi], \quad (13)$$

as well as the pressure  $P(t)$  of load  $M_2$  on the beam, or, which is the same thing, the force compressing the spring. It follows from Formula (13)

$$P(t) = k \delta(\xi) = \beta [w_0(\xi) - f_0(\xi) \sin \pi \xi] P_2. \quad (14)$$

The above method was used for a number of calculations for certain values of parameters  $\alpha$ ,  $\beta$  and  $\mu_2$ .

Figure 3 presents for comparison graphs of variation in the relative dynamic deflections  $f_0(\xi)$  of the midspan (dynamic deflection influence lines), constructed with consideration of spring support for  $\beta = 1$  and without consideration of spring support for  $\beta = \infty$ , for  $M_2 = 1$  and  $\alpha = 0,5$ , as well as the graph of variation in displacement  $w_0(\xi)$  for the same values of  $\mu_2$ ,  $\alpha$  and  $\beta = 1$ .

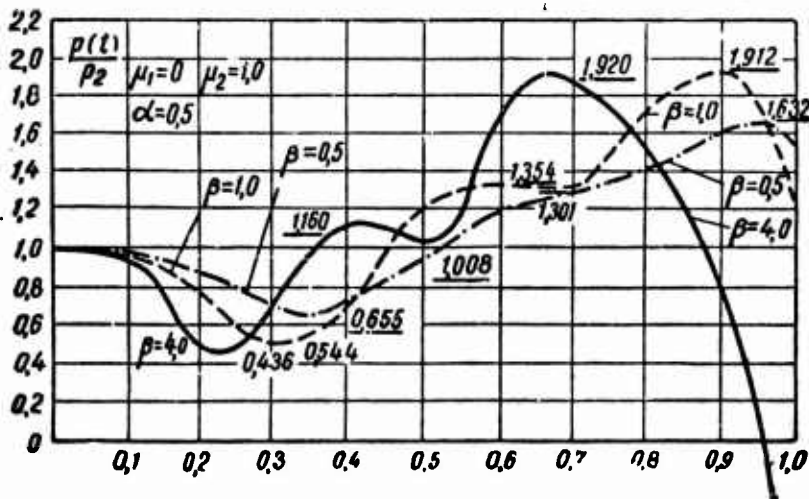


Fig. 5

As follows from Fig. 3, the magnification factor in the presence of a *specified* load-supporting spring is 1.66 instead of the 1.8 in the absence of a spring.

Figure 4 presents the dynamic deflection influence lines for the midspan, constructed for  $\alpha = 0,5$  and  $\mu_2 = 1$  for different values of coefficients  $\beta$ , namely,  $\beta = \infty, 4, 2, 1$  and  $0,5$ .

The table gives the values of the greatest ordinates of these influence lines or, which is the same thing, of magnification factors  $1+\mu$  for the same values of parameters  $\mu$ ,  $\alpha$  and  $\beta$ .

$\beta$	$\infty$	4	1	0,5
$1+\mu$	1,8	1,76	1,66	1,37

The graphs of Fig. 4 and the table give a sufficiently graphic idea about the influence of load-supporting springs on the dynamic effect of a moving load having a mass when moving along a smooth beam.

Figure 5 presents graphs of variations in the pressure of the moving load on the beam or, which is the same thing, of the force compressing the spring during the time interval when the load moves along the beam.

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### Transliterated Symbols

- |    |                                     |
|----|-------------------------------------|
| 71 | ст = st = staticheskiy = static     |
| 73 | балки = balki = balki = of the beam |
| 74 | кр = kr = kriticheskiy = critical   |

# APPLICATION OF THE VARIABLE TIME-SCALE METHOD TO THE SOLUTION OF PROBLEMS OF THE DYNAMIC EFFECT OF A MOVING LOAD ON STRUCTURES

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Problems of the dynamic effect of a moving load on structures are one of the oldest problems of structural mechanics [1]. First these problems were solved without consideration of the masses of the moving load. During the last forty years, starting with the work by Saller [2], a large number of publications has appeared in which the inertia of the moving mass is considered. During the last few years, a great deal of attention was paid to this problem by V.V. Bolotin [3, 4, 5], A.B. Morgayevskiy [6, 7, 8] and S.I. Konashenko [9, 10, 11, 12]. In addition to theoretical work, experimental studies of structures were also performed [13, 14, 15].

The present article presents an approximate method for solving problems of structure vibrations taking into account the mass of the moving load for slowly varying parameters of the system. It is a particular case of the more general method of the variable time scale, developed for nonlinear vibrations [16, 17, 18]. The substance of this method consists in the fact that, by transformation of the time scale, a differential equation with variable coefficients is reduced to the well-studied equation with constant coefficients.

## §1. Substance of the Method

If we consider only the first (fundamental) mode of vibrations of the system and disregard the higher modes, which is usually done in the applied theory of vibrations, then the overwhelming majority of problems of structure vibrations due to a moving load with consideration of its mass can be reduced to a single second-order equation with variable coefficients

$$\ddot{y}(t) + N(t) \dot{y}(t) + T(t) y(t) = F(t). \quad (1)$$

Here and henceforth, the dots denote differentiation with respect to time. Equation (1) cannot always be solved in terms of elementary functions. For this reason, we shall seek its solution on the following considerations. We shall assume that its solution is the solution of a differential equation such which, on the one hand, allows an exact solution in elementary functions, while, on the other hand, it is close to starting equation (1). We shall term

this equation the replacing equation. This approach was used successfully before [9] for finding an approximate solution of one particular case of Eq. (1), with an entirely different method used for comparing the coefficients of the starting and replacing equations.

We shall seek the replacing equation, assuming that it is possible to transform it into a linear equation with constant coefficients, i.e.,

$$\frac{d^2 z(\gamma)}{d\gamma^2} + z(\gamma) = H(\gamma). \quad (2)$$

Using the idea of a variable time scale [16, 17, 18], we substitute

$$z(\gamma) = y(t) \Phi(t), \quad \gamma = \varphi(t), \quad (3)$$

where  $\Phi(t)$  and  $\varphi(t)$  are as yet unknown functions of time. Differentiating the first condition in (3) twice with respect to  $\gamma$ , and using the second substitution of (3), we get

$$\begin{aligned} \frac{dz}{d\gamma} &= \frac{dz}{dt} \cdot \frac{dt}{d\gamma} = (\dot{y} \Phi + y \dot{\Phi}) \frac{1}{\dot{\varphi}}; \\ \frac{d^2 z}{d\gamma^2} &= \frac{d}{d\gamma} \left( \frac{dz}{d\gamma} \right) = \frac{d}{dt} \left( \frac{dz}{d\gamma} \right) \frac{dt}{d\gamma} = \\ &= \frac{1}{\dot{\varphi}^3} [(\ddot{y} \Phi + 2 \dot{\Phi} \dot{y} + y \ddot{\Phi}) \dot{\varphi} - \ddot{\varphi} (\dot{y} \Phi + y \dot{\Phi})]. \end{aligned}$$

Substituting the last expression, as well as the first of Conditions (3) into Eq. (2) and performing simple transformations, we get the replacing equation

$$\ddot{y} + \left( 2 \frac{\dot{\Phi}}{\Phi} - \frac{\ddot{\varphi}}{\dot{\varphi}} \right) \dot{y} + \left( \frac{\ddot{\Phi}}{\Phi} + \dot{\varphi}^2 - \frac{\ddot{\varphi} \dot{\Phi}}{\dot{\varphi} \Phi} \right) y = H(\gamma) \frac{\dot{\varphi}^3}{\Phi}. \quad (4)$$

It is clear from a comparison of Eqs. (1) and (4) that the solutions of these equations will be identical if their coefficients and right-hand sides are equal, i.e.,

$$N(t) = 2 \frac{\dot{\Phi}}{\Phi} - \frac{\ddot{\varphi}}{\dot{\varphi}}; \quad (5)$$

$$T(t) = \frac{\ddot{\Phi}}{\Phi} + \dot{\varphi}^2 - \frac{\ddot{\varphi} \dot{\Phi}}{\dot{\varphi} \Phi}; \quad (6)$$

$$F(t) = H \frac{\dot{\varphi}^3}{\Phi}. \quad (7)$$

From these three conditions we must find the three unknown functions  $\Phi(t)$ ,  $\varphi(t)$  and  $H(\gamma)$ . From Equality (7) we have

$$H(\gamma) = \frac{\Phi}{\dot{\varphi}^3} F(t). \quad (8)$$

The first two conditions (5) and (6) are a system of two equations, of which (6) is nonlinear. It is impossible to obtain an exact solution of this system in elementary functions.

It is possible to obtain an approximate solution by taking into account the fact that for practical problems functions  $N(t)$  and  $T(t)$  vary slowly. A similar assumption was used in [5].

It will be shown below that in this case functions  $\Phi(t)$  and  $\varphi(t)$  will also be slowly varying functions. Consequently  $\dot{\Phi}(t) \approx 0$  and  $\dot{\varphi}(t) \approx 0$ , and

$$\frac{\ddot{\Phi}}{\Phi} - \frac{\ddot{\varphi}\dot{\Phi}}{\dot{\varphi}\Phi} \ll \dot{\varphi}^2.$$

Disregarding second-order infinitesimals, Condition (6) can be reduced to the form

$$T(t) = \dot{\varphi}^2(t). \quad (9)$$

Integrating it, we get

$$\varphi(t) = \int \sqrt{T(t)} dt + C_1. \quad (10)$$

Differentiating Equality (9) with respect to time and substituting the value of  $\ddot{\varphi}$  and  $\dot{\varphi}^2$  according to Formula (9) into Condition (5), we find

$$N(t) = 2 \frac{\dot{\Phi}}{\Phi} - \frac{\ddot{T}}{2T}.$$

Integrating this equation and performing simple transformations, we get

$$\Phi(t) = \left[ \sqrt{T(t)} e^{\lambda(t)} \right]^{\frac{1}{2}}, \quad (11)$$

where

$$\lambda(t) = \int N(t) dt + C_2. \quad (12)$$

In Formulas (10) and (12),  $C_1$  and  $C_2$  are arbitrary constants of integration.

As is known, the solution of Eq. (2) has the form

$$z(\gamma) = A \cos \gamma + B \sin \gamma + \int_{\gamma_0}^{\gamma} H(u) \sin(\gamma - u) du. \quad (13)$$

Here  $A$  and  $B$  are arbitrary constants and, in accordance with the second of Conditions (3), it is denoted as  $u = \varphi(\tau)$ . Substituting this expression and its differential, as well as Equality (8) into Solution (13) and using Conditions (3) for returning to the old variables, we get a solution of replacing Eq. (4) in the form

$$u(t) = \frac{1}{\Phi(t)} \left\{ A \cos \varphi(t) + B \sin \varphi(t) + \int_0^t \frac{\Phi(\tau)}{\dot{\varphi}(\tau)} F(\tau) \sin[\varphi(t) - \varphi(\tau)] d\tau \right\}. \quad (14)$$

The limits of integration are here determined by the interval of variation of variable  $\tau$ , namely  $0 \leq \tau \leq t$ .

Since Solution (14) contains the difference of functions  $\varphi(t) - \varphi(\tau)$ , then constant  $C_1$  in Formula (10) can be taken arbitrarily. Assuming it equal to

$$C_1 = - \int \sqrt{T(t)} dt \Big|_{t=0},$$

it becomes possible to represent Expression (10) thus

$$\varphi(t) = \int_0^t \sqrt{T(\tau)} d\tau. \quad (15)$$

Similarly, it follows from Formulas (11), (12) and (14) that constant  $C_2$  in Eq. (12) can be also chosen arbitrarily. Assuming it equal to

$$C_2 = - \int N(t) dt \Big|_{t=0},$$

we write Formula (12) in the form

$$\lambda(t) = \int_0^t N(\tau) d\tau. \quad (16)$$

We determine arbitrary constants  $A$  and  $B$  contained in Solution (14) from the initial conditions, which for problems of the effect of a moving load on a structure are always zero, i.e.,  $t=0, y=0, \dot{y}=0$ .

Substituting these conditions into Solution (14) and into its first time derivative

$$\dot{y}\Psi + y\dot{\Psi} = \left\{ -A \sin \varphi + B \cos \varphi + \int_0^t \frac{\Phi(\tau)}{\dot{\varphi}(\tau)} F(\tau) \cos [\varphi(t) - \varphi(\tau)] d\tau \right\} \dot{\varphi}(t),$$

we get the homogeneous system

$$A \cos \varphi(0) + B \sin \varphi(0) = 0; \quad -A \sin \varphi(0) + B \cos \varphi(0) = 0.$$

Since the determinant of this system is not zero, i.e.,

$$\begin{vmatrix} \cos \varphi(0) & \sin \varphi(0) \\ -\sin \varphi(0) & \cos \varphi(0) \end{vmatrix} = \cos^2 \varphi(0) + \sin^2 \varphi(0) = 1,$$

then  $A=B=0$ . Consequently, Solution (14) is reduced to the form

$$y(t) = \frac{1}{\Phi(t)} \int_0^t \frac{\Phi(\tau)}{\dot{\varphi}(\tau)} F(\tau) \sin [\varphi(t) - \varphi(\tau)] d\tau.$$

This formula can be represented in still another form

$$y(t) = \frac{1}{\Phi(t)} [f_1(t) \sin \varphi(t) - f_2(t) \cos \varphi(t)], \quad (17)$$

where

$$\left. \begin{aligned} f_1(t) &= \int_0^t \frac{e^{\lambda(\tau)}}{\Phi(\tau)} F(\tau) \cos \varphi(\tau) d\tau; \\ f_2(t) &= \int_0^t \frac{e^{\lambda(\tau)}}{\Phi(\tau)} F(\tau) \sin \varphi(\tau) d\tau. \end{aligned} \right\} \quad (18)$$

Use is made here of the equality

$$\frac{\Phi(t)}{\dot{\Phi}(t)} = \frac{e^{\lambda(t)}}{\Phi(t)},$$

which follows from Formulas (9) and (11).

We now pass on to specific problems, illustrating the above method.

## §2. Problem of a Load Moving on a Beam

Let a load with weight  $P$  and mass  $M_k$  be placed on a beam (Fig. 1) at time  $t = 0$ , and let the load then move along the beam at constant speed  $v$ .

To solve the problem approximately, it is permissible [1-12] to disregard the effect of higher modes of vibrations and to consider the beam as a system with one degree of freedom (Fig. 1).

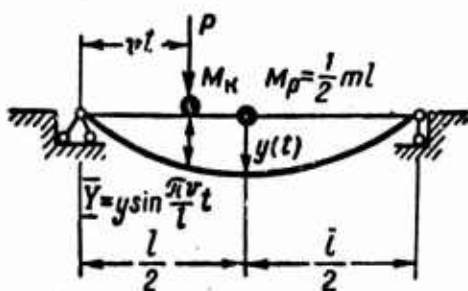


Fig. 1

The mass of the beam reduced to the midspan (without consideration of the mass of the load) will, as is known, be  $M_p = \frac{ml}{2}$ , where  $m$  is the mass per unit length and  $l$  is the beam span, if we take as the approximating curve for the elastic curve of the beam the sinusoid

$$Y(x, t) = y(t) \sin \frac{\pi x}{l}, \quad (19)$$

where  $y(t)$  is the beam midspan deflection.

According to the d'Alembert principle, we can write for the displacement of the beam midspan the expression

$$y(t) = (P + I) \delta_{1p} - M_p \ddot{y}(t) \delta_{11}, \quad (20)$$

where  $\delta_{11}$  is the influence coefficient for displacements, equal to

$$\delta_{11} = \frac{l^3}{48 EJ} \approx \frac{2l^3}{\pi^4 EJ}. \quad (21)$$

$EJ$  is the rigidity of the beam;  
 $\delta_{1p}$  is the influence coefficient for displacements, which  
 can be approximated by the sinusoid

$$\delta_{1p} \approx \delta_{11} \sin \frac{\pi}{l} vt \quad (x = vt), \quad (22)$$

$I$  is the inertia force of the moving load which, according  
 to d'Alembert's principle is

$$I = -M_k \left( \frac{d^2 \bar{Y}}{dt^2} \right)_{x=vt}. \quad (23)$$

Since function of deflections (19) is a function of two variables  $x$  and  $t$ , its total second derivative will consist of three terms

$$\frac{d^2 \bar{Y}}{dt^2} = \frac{\partial^2 \bar{Y}}{\partial t^2} + 2v \frac{\partial^2 \bar{Y}}{\partial x \partial t} + v^2 \frac{\partial^2 \bar{Y}}{\partial x^2}; \quad \left( v = \frac{dx}{dt} \right), \quad (24)$$

which are called the translational, Coriolis and relative accelerations.<sup>1</sup>

Introducing Formula (19) into Eq. (24), and then into Eq. (23), we get

$$I = -M_k [\ddot{y}(t) \sin et + 2\dot{y}(t) e \cos et - y(t) e^2 \sin et],$$

where

$$e = \frac{\pi v}{l}.$$

Substituting this formula into Eq. (20) and referring to Equalities (21) and (22), after elementary transformations we find

$$(M_p + M_k \sin^2 et) \ddot{y} + M_k e \dot{y} \sin 2et + \\ + \left( \frac{\pi^2 EJ}{2l^3} - M_k e^2 \sin^2 et \right) y = P \sin et.$$

Dividing both parts of this equation by  $M_p = \frac{ml}{2}$  and introducing the dimensionless parameter  $\beta = M_k/ml$ , we get

$$(1 + 2\beta \sin^2 et) \ddot{y} + 2e\beta \dot{y} \sin 2et + (\theta^2 - 2e^2 \beta \sin^2 et) y = \frac{P}{M_p} \sin et, \quad (25)$$

where  $\theta = \frac{\pi^2}{l^2} \sqrt{\frac{EJ}{m}}$  is the frequency of the principal mode of free vibrations of the beam without a weight.

This equation was set up by the same method as in [2, 19, 20], where only the translational inertia force was taken into account. Equation (25) is a corollary of the more general Bolotin-Inglis equation [3, 4, 21] and can be obtained in the first approximation from this general equation [3-5]. Equation (25) was for the first time obtained in [9], where it was set up by the method of gen-

eralized coordinates. This reference also called attention to the fact that the coefficients in Eq. (25) are related by a certain relationship, namely, if we denote

$$p(t) = 1 + 2\beta \sin^2 et; \alpha = \frac{v}{v_{kr}}; v_{kr} = \frac{l\theta}{\pi}; S = 1 - \alpha^2 \beta, \quad (26)$$

where  $v_{kr}$  is the critical velocity with which a constant force moves along a beam, then Eq. (25) can be written as

$$p \ddot{y} + \dot{p} \dot{y} + \left( \frac{1}{4} \ddot{p} + S \theta^2 \right) y = \frac{P}{M_p} \sin et. \quad (27)$$

This equation is reduced easily to form (1). In the case under consideration the coefficients and the right-hand side are

$$N(t) = \frac{\dot{p}(t)}{p(t)}; T(t) = \frac{1}{p(t)} \left[ \frac{1}{4} \ddot{p}(t) + \theta^2 S \right], \quad (28)$$

$$F(t) = \frac{P}{M_p} \frac{\sin et}{1 + 2\beta \sin^2 et}. \quad (29)$$

From Formula (16), taking into account the first notation in (26), we have

$$\lambda(t) = \int_0^t N(\tau) d\tau = \int_0^t \frac{1}{p} \cdot \frac{dp}{d\tau} d\tau = \int_0^t \frac{dp(\tau)}{p(\tau)} = \ln p(t).$$

Consequently,

$$e^{\lambda(t)} = p(t). \quad (30)$$

For middle- and large-span structures function  $p(t)$ , which represents the mass of the system reduced to midspan, varies slowly. This makes it possible to disregard in the second of Formulas (28) the value of  $\frac{1}{4} \ddot{p}(t)$  as compared with  $\theta^2 S$ , i.e., to assume that

$$T(t) = \frac{\theta^2 S}{p(t)}. \quad (31)$$

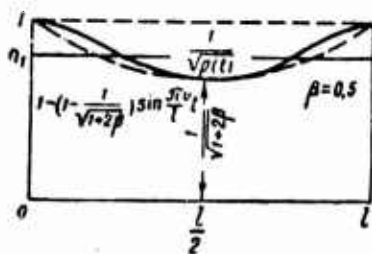


Fig. 2

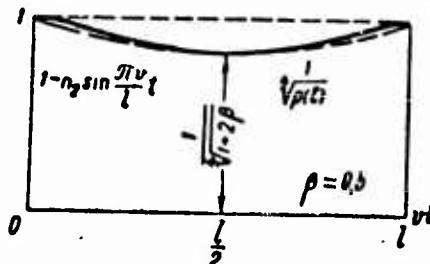


Fig. 3

Using Formulas (15) and (31), we get for the function of the phase angle the expression

$$\varphi(t) = \theta \sqrt{S} \int_0^t \frac{d\tau}{\sqrt{\rho(\tau)}} = \theta \sqrt{S} \int_0^t \frac{d\tau}{\sqrt{1+2\beta \sin^2 e\tau}}. \quad (32)$$

Then, according to Formulas (11), (30) and (31), we have

$$\Phi(t) = \left[ \rho(t) \sqrt{\frac{\theta^2 S}{\rho(t)}} \right]^{\frac{1}{2}} = \rho^{\frac{1}{4}} \sqrt{\theta^2 S}. \quad (33)$$

In the case of free vibrations, setting in Solution (14)  $F(\tau) \equiv 0$ , we find

$$y(t) = \frac{1}{\Phi(t)} [A \cos \varphi(t) + B \sin \varphi(t)].$$

Substituting here Expressions (32) and (33), we get

$$y(t) = \theta^{-\frac{1}{2}} S^{-\frac{1}{4}} \rho(t)^{-\frac{1}{4}} \left[ A \cos \theta \sqrt{S} \int_0^t \frac{d\tau}{\sqrt{\rho(\tau)}} + B \sin \theta \sqrt{S} \int_0^t \frac{d\tau}{\sqrt{\rho(\tau)}} \right].$$

Including  $\theta^{-\frac{1}{2}} S^{-\frac{1}{4}}$  into the values of constants  $A$  and  $B$ , we shall arrive at the expression obtained previously in [9] in another way.

The integral in Formula (32) cannot be expressed in terms of elementary functions. For this reason, we shall find its approximate value, by linearizing the phase-angle function from the condition of equality of areas (Fig. 2)

$$\int_0^t \frac{d(vt)}{\sqrt{1+2\beta \sin^2 et}} \approx \int_0^t \left[ 1 - \left( 1 - \frac{1}{\sqrt{1+2\beta}} \right) \sin^2 et \right] d(vt) = n_1 t,$$

where

$$n_1 = 1 - \frac{2}{\pi} \left( 1 - \frac{1}{\sqrt{1+2\beta}} \right). \quad (34)$$

Consequently, according to Formula (32), we have

$$\varphi(t) \approx \theta \sqrt{S} \int_0^t n_1 d\tau = \theta \sqrt{S} n_1 t = \psi t; \quad (\psi = \theta \sqrt{S} n_1). \quad (35)$$

Introducing into Equalities (18) Expressions (29), (30), (33) and (35), we find

$$f_1(t) = \frac{P}{M_p \sqrt[4]{\theta^3 S}} \int_0^t p^{-\frac{1}{4}}(\tau) \sin e\tau \cos \psi\tau d\tau,$$

$$f_2(t) = \frac{P}{M_p \sqrt[4]{\theta^3 S}} \int_0^t p(\tau)^{-\frac{1}{4}} \sin e\tau \sin \psi\tau d\tau.$$

Substituting here the approximate expression (Fig. 3)

$$p(t)^{-\frac{1}{4}} = \frac{1}{\sqrt{1+2\beta \sin^2 et}} \approx 1 - n_2 \sin et; \quad n_2 = 1 - \frac{1}{\sqrt{1+2\beta}} \quad (36)$$

and integrating, we get

$$f_1(t) = \frac{P}{M_p \sqrt[4]{\theta^3 S}} \left[ \frac{1}{2\eta_1} (1 - \cos \eta_1 t) + \frac{1}{2\eta_3} (1 - \cos \eta_3 t) - \right. \\ \left. - \frac{n_2}{4} \left( \frac{2 \sin \psi t}{\psi} - \frac{1}{\eta_3} \sin \eta_3 t - \frac{1}{\eta_4} \sin \eta_4 t \right) \right],$$

$$f_2(t) = \frac{P}{M_p \sqrt[4]{\theta^3 S}} \left\{ \frac{1}{2\eta_1} \sin \eta_1 t - \frac{1}{2\eta_3} \sin \eta_3 t + \frac{n_2}{4} \left[ \frac{2}{\psi} (\cos \psi t - 1) + \right. \right. \\ \left. \left. + \frac{1}{\eta_3} (\cos \eta_3 t - 1) - \frac{1}{\eta_4} (\cos \eta_4 t - 1) \right] \right\},$$

where

$$\eta_1 = e - \psi = \frac{\pi v}{l} - \theta n_1 \sqrt{S}; \quad \eta_3 = 2e - \psi = 2 \frac{\pi v}{l} - \theta n_1 \sqrt{S};$$

$$\eta_2 = e + \psi = \frac{\pi v}{l} + \theta n_1 \sqrt{S}; \quad \eta_4 = 2e + \psi = 2 \frac{\pi v}{l} + \theta n_1 \sqrt{S}.$$

Introducing expressions for  $f_1(t)$  and  $f_2(t)$ , as well as Formulas (33), (35) and (36) into Solution (17), and then transforming, we get

$$y(t) = \frac{P}{M_p \theta \sqrt{S}} (1 - n_2 \sin et) \left[ \frac{e \sin \psi t - \psi \sin et}{e^2 - \psi^2} + \right. \\ \left. + \frac{n_2 \left( \psi \sin^2 et - 4e^2 \sin^2 \frac{\psi}{2} t \right)}{\psi (e^2 - \psi^2)} \right]. \quad (37)$$

Substituting here  $n_2=0$ ,  $\lambda=1$  and  $\psi=\theta$ , we obtain the known [19] solution for the problem of motion of a concentrated force along a beam

$$y(t) = \frac{P}{M_p (\theta^2 - e^2)} \left( \sin et - \frac{e}{\theta} \sin \theta t \right).$$

Multiplying the numerator and denominator of Formula (37) by  $\delta_{11}$  and taking into account Equalities (26) and  $y_{cr} = P \delta_{11}$ ;  $\frac{1}{M_p \delta_{11}} = \theta^2$ , and also changing to a new variable  $\frac{v t}{l} = \xi$ , we get the following expression for the magnification factor for a moving load

$$(1 + \mu) = \frac{y(\xi)}{y_{cr}} = \frac{1}{\alpha \sqrt{S}} (1 - n_2 \sin \pi \xi) \times$$

$$\times \left\{ \frac{\sin \frac{\sqrt{S}}{\alpha} n_1 \pi \xi - \frac{\sqrt{S}}{\alpha} n_1 \sin \pi \xi}{1 - \left( n_1 \frac{\sqrt{S}}{\alpha} \right)^2} + \right.$$

$$\left. + \frac{n_2 \left[ \left( n_1 \frac{\sqrt{S}}{\alpha} \right)^2 \sin^2 \pi \xi - 4 \sin^2 \frac{\sqrt{S}}{\alpha} \cdot \frac{n_1}{2} \pi \xi \right]}{4 n_1 \frac{\sqrt{S}}{\alpha} - \left( n_1 \frac{\sqrt{S}}{\alpha} \right)^2} \right\}. \quad (38)$$

It follows from analyzing Formula (38) that as  $\alpha \rightarrow n_1 \sqrt{S}$  the first term becomes an indeterminacy of the type 0/0. Solving this indeterminacy by l'Hospital's rule, we get an expression for the magnification factor, in this case

$$(1 + \mu) = \frac{n_1}{2} (1 - n_2 \sin \pi \xi) (\sin \pi \xi - \pi \xi \cos \pi \xi +$$

$$+ \frac{2}{3} n_2 \sin^3 \pi \xi - \frac{8}{3} n_2 \sin^2 \frac{1}{2} \pi \xi). \quad (39)$$

To estimate the accuracy of the suggested method, the results of calculations using Formulas (38) and (39) were compared with results of numerical integration of Eq. (25), which was done in [8]. The results are displayed in Figs. 4 and 5, which show graphs of magnification factors for mass ratio  $\beta = 0.5$  (Fig. 4) and  $\beta = 1$  (Fig. 5) for different velocity ratios  $(v/v_{kr}^*)$  given on the graphs by circles. The dashed lines show the results of numerical integration, which can be regarded as exact, while the solid lines present results obtained from Formulas (38) and (39), where

$$\alpha = \frac{v}{v_{kp}^*} \cdot \frac{1}{\sqrt{1 + 2\beta}}.$$

Here  $v_{kr}^*$  is the critical velocity of the load, which is [8]

$$v_{kp}^* = \frac{v_{kp}}{\sqrt{1 + 2\beta}}.$$

It can be seen by comparing graphs in Figs. 4 and 5 that the magnification factors have been determined with a quite poor accuracy. This is due to the fact that function  $p(t)$  was assumed to be a slowly varying function and the value of  $\ddot{p}(t)$  was disregarded on this basis. It is natural that a solution of Eq. (27) was obtained which is valid for small velocities of a load moving along a beam.

We now solve Eq. (27) without disregarding the value of  $\ddot{p}(t)$ . The coefficient of  $y$  in Eq. (27) can be represented still in another manner, i.e.,

$$\frac{1}{4} \ddot{p}(t) + S\theta^2 = \theta^2 (1 - 2\alpha^2 \beta \sin^2 \pi \xi).$$

Then according to Formulas (15) and (28), we have

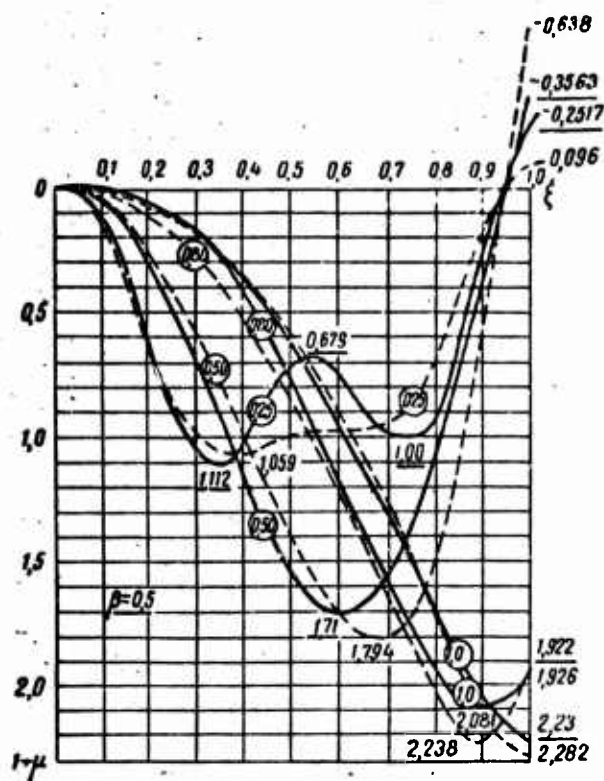


Fig. 4

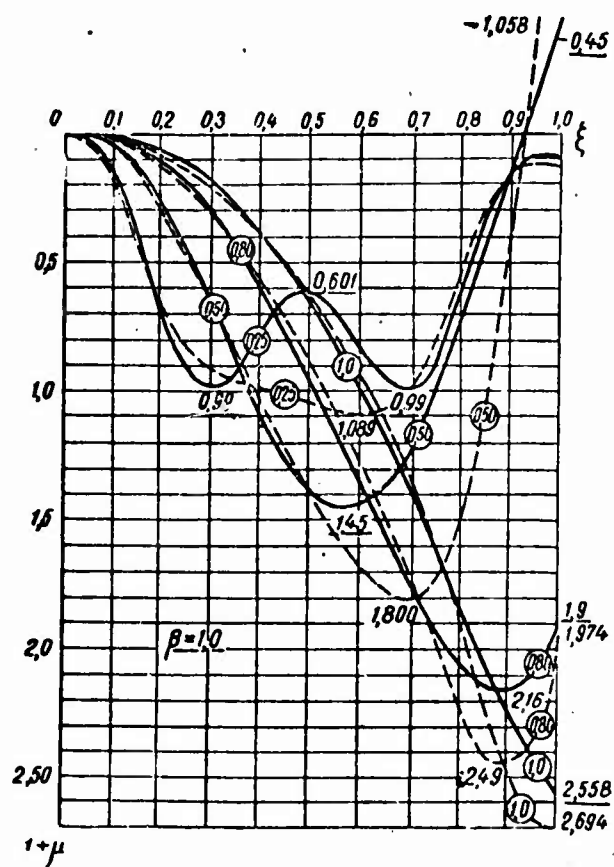


Fig. 5

$$\varphi(t) = \int_0^t \sqrt{T(\tau)} d\tau = \theta \int_0^t \frac{\sqrt{1 - 2\alpha^2 \beta \sin^2 e\tau}}{\sqrt{1 + 2\beta \sin^2 e\tau}} d\tau.$$

The denominator of this integrand was approximated earlier (34) by a straight line (coefficient  $n_1$ ). The numerator is approximated similarly

$$\int_0^t \sqrt{1 - 2\alpha^2 \beta \sin^2 \frac{\pi}{l} vt} d(vt) \approx \int_0^t [1 - (1 - \sqrt{1 - 2\alpha^2 \beta}) \times \sin \frac{\pi}{l} vt] d(vt) = n_0 t, \quad (40)$$

where

$$n_0 = 1 - \frac{2}{\pi} (1 - \sqrt{1 - 2\alpha^2 \beta}).$$

Then we get for the phase angle function

$$\varphi(t) = \theta \int_0^t n_0 n_1 d\tau = \psi_1 t; \quad (\psi_1 = \theta n_0 n_1). \quad (41)$$

It would now have been possible to write on the basis of (37) a more exact expression for the midspan displacement, replacing there  $\sqrt{s}$  by  $n_0$  and  $\psi$  by  $\psi_1$ . However, this formula can be somewhat simplified if  $p(t)^{-\frac{1}{4}}$  (Fig. 3) is also approximated by a straight line from the equal-size condition

$$\int_0^t p^{-\frac{1}{4}}(vt) d(vt) = \int_0^t \left(p^{-\frac{1}{2}}\right)^{\frac{1}{2}} d(vt) \approx \sqrt{n_1} t, \quad (42)$$

where

$$n_1 = 1 - \frac{2}{\pi} \left(1 - \frac{1}{\sqrt{1 + 2\beta}}\right);$$

then, according to Formulas (11), (28), (30), (40) and (42), we have

$$\Phi(t) = [1/T(t)e^{\lambda(t)}]^{\frac{1}{2}} \approx \sqrt{\frac{\theta n_0}{n_1}}. \quad (43)$$

Introducing into Equalities (18), Expressions (29), (30), (41) and (42) and integrating, we get

$$f_1(t) = \frac{P \sqrt{n_1}}{2M_p \sqrt{\theta n_0}} \left[ \frac{1 - \cos(e + \psi_1)t}{e + \psi_1} + \frac{1 - \cos(e - \psi_1)t}{e - \psi_1} \right];$$

$$f_2(t) = \frac{P \sqrt{n_1}}{2M_p \sqrt{\theta n_0}} \left[ \frac{\sin(e - \psi_1)t}{e - \psi_1} - \frac{\sin(e + \psi_1)t}{e + \psi_1} \right].$$

Substituting the expressions for  $f_1(t)$  and  $f_2(t)$ , as well as Formulas (41) and (43), into Solution (17) and transforming, we get

$$y(t) = \frac{P n_1}{M_p \theta n_0 (s^2 - \psi_1^2)} [e \sin \psi_1 t - \psi_1 \sin e t]. \quad (44)$$

Changing to the variable  $\xi = \frac{vt}{l}$  for the magnification factor for a moving load, we get the expression

$$(1 + \mu) = \frac{y(\xi)}{y_{cr}} = \frac{n_1}{a n_0 \left[ 1 - \left( \frac{n_0 n_1}{a} \right)^2 \right]} \left[ \sin \frac{n_0 n_1}{a} \pi \xi - \frac{n_0 n_1}{a} \sin \pi \xi \right] \quad (45)$$

In the case of  $a = n_0 n_1$ , eliminating the indeterminacy we have

$$(1 + \mu) = \frac{1}{2 n_0^2} [\sin \pi \xi - \pi \xi \cos \pi \xi]. \quad (46)$$

Due to the additional approximation of function  $p^{-1/4}(t)$  the expressions for the magnification factor have been thus substantially simplified and their appearance is similar to the solution of a second-order equation with constant coefficients.

The question can be raised whether the same result could not be obtained by approximating the coefficients in starting equation (27). Since functions

$$\dot{p}(t) = 2e\beta \sin 2et \text{ and } \ddot{p}(t) = 4e^2\beta \cos 2et$$

have a frequency  $2e$ , then, when approximating from the requirement of equal areas, we get

$$\ddot{p}(t) = 0, \quad \dot{p}(t) = 0.$$

This corresponds to disregarding a part of the relative and the entire Coriolis acceleration. It is natural that this solution will not conform to reality; to obtain a more exact (but more cumbersome) solution it would be necessary to break up function  $p(t)$  into a sufficiently large number of segments and to integrate them with different initial conditions, as this was done in [9].

To determine the accuracy of Formulas (45) and (46), graphs were constructed which are shown in Figs. 6 and 7. The designations in them are the same as in Figs. 4 and 5.

As follows from the graphs, despite the substantial simplification of the formulas, the accuracy with which the maximum values of magnification factors was determined increased perceptibly. This took place because the term containing  $\ddot{p}(t)$  was taken into account.

For small load velocities, the results obtained from Formulas (45) and (46) are compared with the maximum values of magnification factors obtained experimentally in [14]. The results of this comparison are presented in the table.

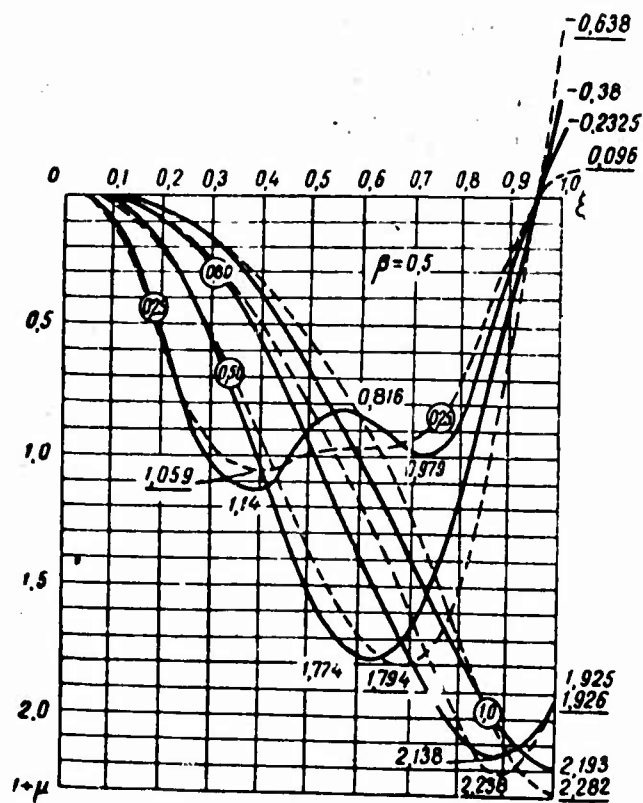


Fig. 6

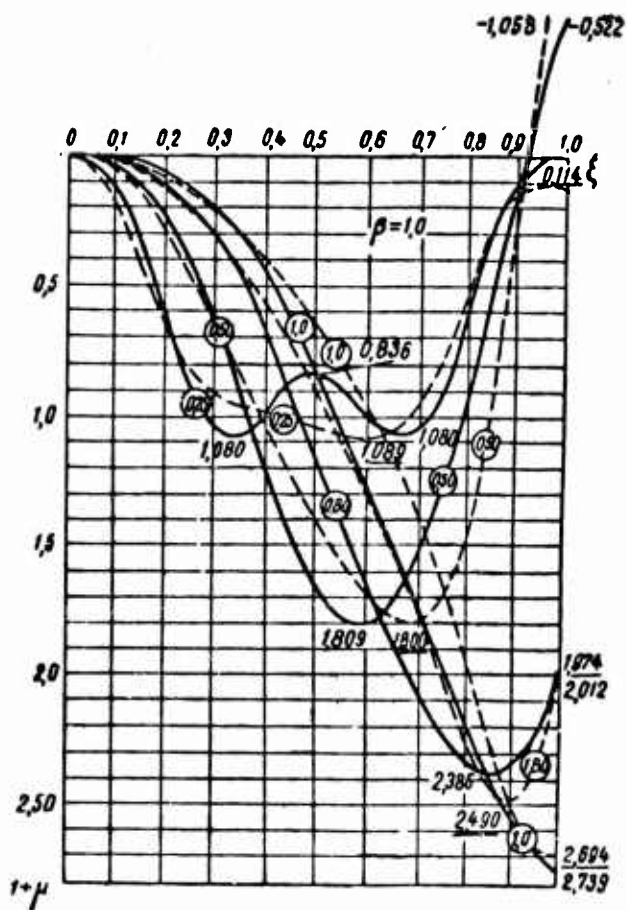


Fig. 7

№ п/п	а	1+μ		№ п/п	а	1-	
		Из опыта В	Теоретиче- ские С			Из опыта В	Теоретиче- ские С
1	0,124	1,14	1,15	6	0,0955	1,10	1,12
2	0,116	1,14	1,17	7	0,0915	1,08	1,11
3	0,11	1,15	1,15	8	0,086	1,05	1,06
4	0,105	1,12	1,16	9	0,0825	1,02	1,06
5	0,101	1,13	1,15	10	0,074	1,04	1,08

A) Ordinal No.; B) experimental; C) theoretical.

As can be seen from the table, the maximum values of magnification factors at low velocities are also determined with a satisfactory accuracy and the average error is ~2.2%.

The accuracy in determining the magnification factors can be improved further by approximating the functions (Figs. 2 and 3) not by one but by several steps.

### §3. Problem of the Motion of a Distributed Load along a Beam

Let a uniformly distributed load  $q$  per unit length with mass  $m_k$  per unit length be placed at time  $t = 0$  on a beam with reduced mass  $M_p = m_p l/2$  and then move with constant velocity  $v$  (Fig. 8).

The differential equation of the beam's midspan vibrations, set up similarly to Eq. (20), has the form

$$y(t) = \int_0^{vt} (q + i) \delta_{1p}(x) dx - M_p \ddot{y}(t) \delta_{11}. \quad (47)$$

Here the equation of the deflections influence line at beam midspan  $\delta_{1p}(x)$  and the displacements influence coefficient  $\delta_{11}$  can be, as before, determined from Formulas (21) and (22).

The inertia force per unit length, according to d'Alembert's principle, is

$$i = -m_k \frac{d^2 \bar{y}}{dt^2} = -m_k \left( \frac{\partial^2 \bar{y}}{\partial t^2} + 2v \frac{\partial^2 \bar{y}}{\partial x \partial t} + v^2 \frac{\partial^2 \bar{y}}{\partial x^2} \right). \quad (48)$$

Here, as before, Sinusoid (19) is taken as the first vibratory mode. Substituting Expressions (19), (22) and (48) into Eq. (47), we get

$$y = \delta_{11} \left\{ \int_0^{vt} \left[ q - m_k \left( \ddot{y} \sin \frac{\pi}{l} x + 2\dot{y} \frac{v\pi}{l} \cos \frac{\pi}{l} x - v^2 y \left( \frac{\pi}{l} \right)^2 \sin \frac{\pi}{l} x \right) \right] \sin \frac{\pi}{l} x dx - M_p \ddot{y} \right\}.$$

Integrating and transforming, we get

$$M_p \left[ 1 + \frac{\beta}{\pi} \left( \epsilon t - \frac{1}{2} \sin 2\epsilon t \right) \right] \ddot{y} + v m_k \dot{y} \sin^2 \epsilon t + \left[ \frac{1}{\delta_{11}} - v^2 \frac{\pi m_k}{2l} \left( \epsilon t - \frac{1}{2} \sin 2\epsilon t \right) \right] y = \frac{ql}{\pi} (1 - \cos \epsilon t). \quad (49)$$

The notation used here is

$$\beta = \frac{m_k}{m_p}; \quad \epsilon = \frac{\pi v}{l}. \quad (50)$$

Dividing Eq. (49) by the coefficient of  $\ddot{y}$ , which is the reduced mass of the system, we arrive at Eq. (1), where

$$\left. \begin{aligned} N(t) &= \frac{2\beta v \sin^2 \epsilon t}{l \left[ 1 + \frac{\beta}{\pi} \rho(t) \right]}; & T(t) &= \frac{\Theta^2 - \frac{\epsilon^2}{\pi} \beta \rho(t)}{1 + \frac{\beta}{\pi} \rho(t)}; \\ F(t) &= \frac{ql(1 - \cos \epsilon t)}{M_p \pi \left[ 1 + \frac{\beta}{\pi} \rho(t) \right]}; & \rho(t) &= \epsilon t - \frac{1}{2} \sin 2\epsilon t. \end{aligned} \right\} \quad (51)$$

Here  $\Theta^2 = 1/M_p \delta_{11}$  is the square of the frequency of free vibrations of a beam without a load.

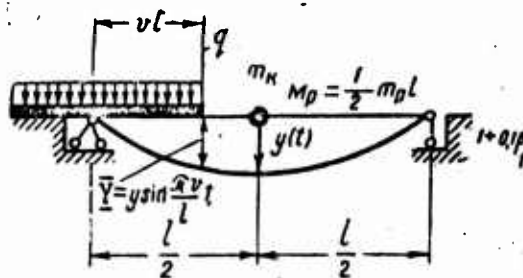


Fig. 8

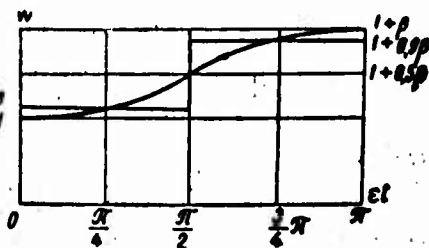


Fig. 9

If we introduce the notation

$$W(t) = 1 + \frac{\beta}{\pi} \rho(t) = 1 + \frac{\beta}{\pi} \left( \epsilon t - \frac{1}{2} \sin 2\epsilon t \right),$$

then, as follows from Expressions (50) and (51), coefficient  $N(t)$  can be represented as

$$N(t) = \frac{\dot{W}(t)}{W(t)}.$$

In accordance with Formula (16) we have

$$\lambda(t) = \int_0^t N(\tau) d\tau = \int_0^t \frac{dW(\tau)}{W(\tau)} = \ln W(t); \quad [\ln W(0) = 0].$$

From this follows that

$$e^{(1)} = W(t) = 1 + \frac{\beta}{\pi} p(t) = 1 + \frac{\beta}{\pi} \left( \epsilon t - \frac{1}{2} \sin 2\epsilon t \right). \quad (52)$$

To simplify the further computations we shall operate in the first approximation by an average value of function  $W$ , which is equal to (Fig. 9)

$$W \approx 1 + \frac{1}{2} \beta. \quad (53)$$

In addition, since  $\epsilon \ll \theta$ , then it is possible to disregard the quantity  $\frac{\epsilon^2}{\pi} \beta p(t)$  as compared with  $\theta^2$ .

Thus, according to Formulas (15) and (51), we find the following linear approximation for the phase angle function  $\varphi(t)$ ,

$$\varphi(t) = \int_0^t \sqrt{T(\tau)} d\tau \approx \int_0^t \sqrt{\frac{\theta^2}{1 + \frac{1}{2} \beta}} d\tau = \psi t. \quad (54)$$

Here

$$\psi = \frac{\theta}{\sqrt{1 + \frac{1}{2} \beta}}. \quad (55)$$

Now by virtue of Formulas (11), (52), (53) and (55) we find

$$\Phi(t) = \sqrt{\psi W(t)} \approx \sqrt{\theta^2 (1 + \beta/2)}. \quad (56)$$

Further, substituting Expressions (51), (52), (53), (55) and (56) into Equalities (18) and integrating, we have

$$\left. \begin{aligned} f_1(t) &= \frac{ql}{\pi M_p \sqrt{\theta^2 (1 + \beta/2)}} \left[ \frac{1}{\psi} \sin \psi t - \frac{1}{2} \left( \frac{1}{\eta_1} \sin \eta_1 t + \frac{1}{\eta_2} \sin \eta_2 t \right) \right]; \\ f_2(t) &= \frac{ql}{\pi M_p \sqrt{\theta^2 (1 + \beta/2)}} \left[ \frac{1}{\psi} (1 - \cos \psi t) + \frac{1}{2} \left( \frac{1}{\eta_1} \cos \eta_1 t + \frac{1}{\eta_2} \cos \eta_2 t - \frac{1}{\eta_1} - \frac{1}{\eta_2} \right) \right], \end{aligned} \right\} \quad (57)$$

where

$$\eta_1 = \psi + \epsilon; \quad \eta_2 = \psi - \epsilon.$$

Substituting these expressions, as well as Formula (56) into Solution (17) and transforming, we get

$$y(t) = \frac{ql}{\pi M_p \theta \sqrt{1 + \beta/2}} \left[ \frac{1}{\psi} (1 - \cos \psi t) + \frac{\psi}{\psi^2 - \epsilon^2} (\cos \psi t - \cos \epsilon t) \right]. \quad (58)$$

Setting here  $\beta=0$  and  $\psi=\theta$ , we arrive at the known [19] solution for a massless distributed load

$$y(t) = \frac{ql}{\pi M_p (\theta^2 - \epsilon^2)} \left[ (1 - \cos \epsilon t) - \left( \frac{\epsilon}{\theta} \right)^2 (1 - \cos \theta t) \right]. \quad (59)$$

Expression (58) can be imparted a similar form, namely

$$y(t) = \frac{ql}{\pi M_p \left(1 + \frac{1}{2}\beta\right) (\psi^2 - \epsilon^2)} \left[ (1 - \cos \epsilon t) - \left(\frac{\epsilon}{\psi}\right)^2 (1 - \cos \psi t) \right]. \quad (60)$$

Multiplying the numerator and denominator of Equalities (59) and (60) by  $\delta_{11}$  and taking into account the formulas

$$\frac{1}{M_p \delta_{11}} = \Theta^2 = \psi^2 \left(1 + \frac{1}{2}\beta\right); \quad \delta_{11} \approx \frac{2l^3}{\pi^4 EJ}; \quad y_{cr} \approx \frac{4ql^4}{\pi^4 EJ}, \quad (61)$$

we get the identical expression for the magnification factors

$$1 + \mu = \frac{y(t)}{y_{cr}} = \frac{1}{2(1 - k^2)} [(1 - \cos \epsilon t) - k^2 (1 - \cos \psi t)], \quad (62)$$

where  $k = \epsilon/\psi$  in the case when the inertia of the moving mass is considered and  $k = \epsilon/\Theta$  when it is not considered ( $\psi = \Theta$ ). Introducing the new variable  $\eta/l = \xi$ , it is possible to write Expression (62) in the form

$$1 + \mu = \frac{1}{2} + \frac{1}{2 \left[1 - \alpha^2 \left(1 + \frac{1}{2}\beta\right)\right]} \left[ \alpha^2 \left(1 + \frac{1}{2}\beta\right) \cos \frac{\pi \xi}{\alpha \sqrt{1 + \frac{1}{2}\beta}} - \cos \pi \xi \right], \quad (63)$$

where  $\alpha$  is determined from Formula (26).

In the second approximation we approximate function  $W(t)$  by two steps (Fig. 9)

$$\begin{aligned} 0 \leq \epsilon t < \frac{\pi}{2}; \quad W_1 &= 1 + \frac{\beta}{2\pi} \left( \frac{\pi}{2} - 1 \right) = 1 + 0.1\beta; \\ \frac{\pi}{2} < \epsilon t \leq \pi; \quad W_2 &= 1 + \frac{\beta}{2\pi} \left( \frac{3}{2}\pi - 1 \right) = 1 + 0.9\beta. \end{aligned}$$

Similarly to the preceding we find

$$\begin{aligned} 0 \leq \epsilon t < \frac{\pi}{2}; \quad \varphi_1(t) &= \psi_1 t; \quad \psi_1 = \frac{\Theta}{\sqrt{1 + 0.1\beta}}; \\ \Phi_1 &= \sqrt{\psi_1 W_1} = \sqrt{\Theta^2 (1 + 0.1\beta)}; \\ \frac{\pi}{2} < \epsilon t \leq \pi; \quad \varphi_2(t) &= \psi_2 t; \quad \psi_2 = \frac{\Theta}{\sqrt{1 + 0.9\beta}}; \\ \Phi_2 &= \sqrt{\psi_2 W_2} = \sqrt{\Theta^2 (1 + 0.9\beta)}. \end{aligned}$$

It is easy to see that as long as  $0 \leq t < \frac{\pi}{2\epsilon}$ , Quadratures (18) are, as before, given by Formulas (57) if  $\psi$  is replaced there by  $\psi_1$  and  $\frac{1}{2}\beta$  is replaced by  $0.1\beta$ .

Consequently, for the case when the load is situated on the first half of the span ( $0 \leq t < \frac{\pi}{2\epsilon}$ ), the midspan displacement and the

magnification factor should be determined from Formulas (58), (60) and (63), replacing there  $\psi$  by  $\psi_1$  and  $\beta/2$  by  $0.1\beta$ , i.e.,

$$(1 + \mu) = \frac{1}{2} + \frac{1}{2[1 - \alpha^2(1 + 0.1\beta)]} \left[ \alpha^2(1 + 0.1\beta) \cos \frac{\pi \xi}{\alpha \sqrt{1 + 0.1\beta}} - \cos \pi \xi \right]. \quad (64)$$

If the load is situated on the second half of the span  $\left(\frac{\pi}{2\epsilon} < t < \frac{\pi}{\epsilon}\right)$  and since the initial conditions will no longer be zero, use should be made of Solution (14) which, by analogy with Formula (17), can be written as

$$y(t) = \frac{1}{\Phi_2(t)} \{ [B + f_1(t)] \sin \varphi_2(t) + [A - f_2(t)] \cos \varphi_2(t) \}, \quad (65)$$

where  $f_1(t)$  and  $f_2(t)$  are determined by Expressions (18) upon replacing the lower limit of integration by  $\pi/2\epsilon$ , while constants  $A$  and  $B$  are found from the initial conditions, obtained from Solution (64) for the first half of the span for  $t = \frac{\pi}{2\epsilon}$ .

Substituting the values of  $A$  and  $B$ , as well as of  $f_1(t)$  and  $f_2(t)$  into Solution (65), we finally get the following for the magnification factor for a load moving on the second half of the beam

$$(1 + \mu) = \frac{1}{2} - \frac{1}{2[1 - \alpha^2(1 + 0.9\beta)]} \cos \pi \xi + \frac{1}{2[1 - \alpha^2(1 + 0.1\beta)]} \left\{ \alpha^2(1 + 0.1\beta) \cos \frac{\pi}{2\alpha \sqrt{1 + 0.1\beta}} \times \right. \\ \times \cos \frac{\pi}{\alpha \sqrt{1 + 0.9\beta}} (\xi - 0.5) - \alpha \sqrt{1 + 0.9\beta} \left[ \frac{0.8\alpha^3\beta}{1 - \alpha^2(1 + 0.9\beta)} + \right. \\ \left. + \alpha \sqrt{1 + 0.1\beta} \sin \frac{\pi}{2\alpha \sqrt{1 + 0.1\beta}} \right] \sin \frac{\pi}{\alpha \sqrt{1 + 0.9\beta}} (\xi - 0.5) \left. \right\}. \quad (66)$$

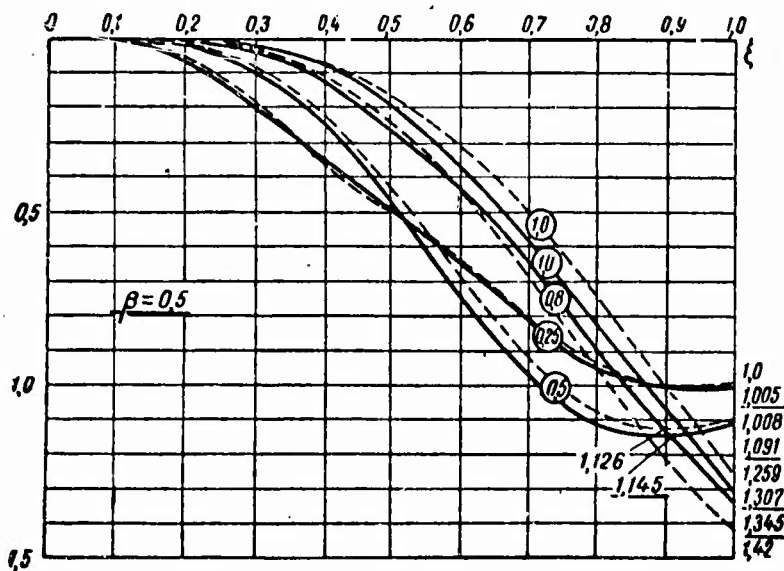


Fig. 10

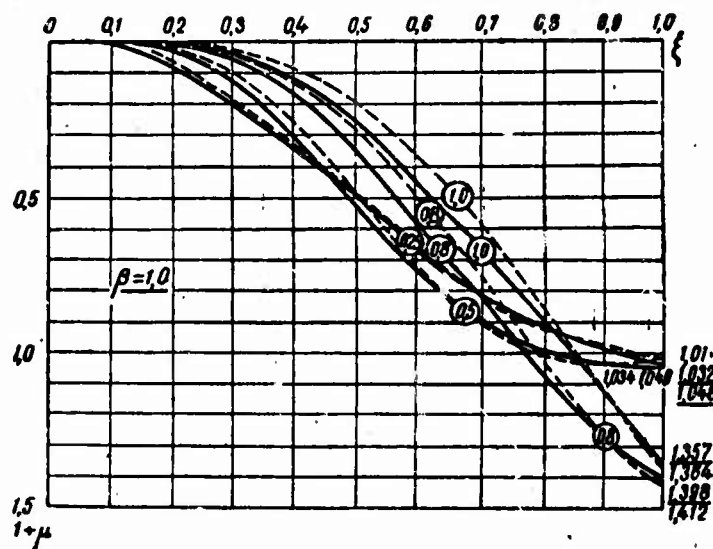


Fig. 11

To compare the accuracy of the results, Figs. 10 and 11 present graphs of magnification factors calculated from Formulas (63), (64) and (66) for a ratio of masses per unit length of  $\beta = 0.5$  (Fig. 10) and  $\beta = 1$  (Fig. 11) and for different ratios  $v/v^*_{kr}$ , where  $v^*_{kr}$  by analogy with the problem of a moving load is given by the expression  $v^*_{kr} = \frac{v_{kp}}{\sqrt{1+2\beta}}$ . The dashed lines on the graphs show the magnification factors calculated from the first-approximation formula (63), while the solid lines denote the factors calculated from the second approximation (64)-(65). The circled numbers denote ratios  $v/v^*_{kr}$ .

Comparison of the graphs shows that the second approximation is close to the first. This means that the second approximation is sufficiently accurate.

In conclusion it should be noted that the above approach yields a simple and sufficiently accurate analytic solution of problems of dynamic effect of a moving load on structures for an arbitrary disturbance.

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#### Footnote

- 83      <sup>1</sup>The components of inertia force (23) have analogous designations.

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# Transliterated Symbols

84	кp = kr = kriticheskiy = critical
86	ст = st = staticheskiy = static

# THERMOELASTIC AND THERMOPLASTIC VIBRATIONS OF BARS SYSTEMS AND PLATES<sup>1</sup>

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## §1. Statement of the Problem

A sudden change in the temperature of elements of elastic systems brings about a rapid increase in the structure's deformation, with the result that vibrations which have a substantial effect on the stressed state of the entire system are excited. In this case, temperature stresses are produced even in statically determinate systems. Of substantial significance for estimating the effect of sudden heating is the problem of heat propagation inside elements of elastic systems.

In a number of problems of practical importance the problem of thermal conductivity was solved by Academician A.N. Krylov [1]; the method he has developed makes it possible to find the principal parameters of the temperature field for various simple computational systems. Consideration of inertia forces attendant to sudden heating of an elastic half space was implemented by V.I. Danilovskaya [2, 3], whose work was followed by a large number of Soviet and foreign scientists.

New problems were examined and solved by A.A. Il'yushin, N.A. Slezkin, P.M. Ogibalov, P.M. Malinin, V.V. Novatskiy, A.P. Sinitsyn, R. Chadwick, I.N. Sneddon, W.A. Bailey, H. Parkus, E. Melan, I.H. Weiner, I.E. Michaels, E. Sternberg, R. Muki, N. Hoff, and others. As a result of this, the general theory of thermoelastic vibrations is at present sufficiently well developed and it is possible to use it for engineering calculations of complex computational schemes of bar systems subjected to thermal shock. A problem worked out much less is that of the effect on the stressed state of a system of temperature-induced changes in physical constants, i.e., linear expansion coefficient, elastic modulus and the Poisson ratio of elastic system. Of substance, in particular for new materials, are also changes in creep and ductility properties. Up to lately, these factors were estimated without taking inertia forces into account; the problem then was solved in the linear approximation [4, 5]. Another substantial complication of the problem is due to taking into account the interrelationships between the thermal conductivity and elasticity equations [6, 7, 8].

The present paper presents a method of designing bar systems and plates for thermal shock, taking into account variations in the physical parameters of the system which take place due to temperature. This method of calculation is a development of a method worked out by the present author previously and presented in [9, 10, 11]. Elastic bar systems will be considered as systems with several degrees of freedom, and the nonlinear vibrations will be analyzed by methods presented in [12, 13].

The problem may become nonlinear for various reasons. For example, the thermal shock may produce in individual elements of the system stresses which go past the elastic limits, with formation of plastic domains and attendant nonlinear vibrations. Nonlinear vibrations of a system with variable physical parameters will be considered by a method of stepped approximation, which yields satisfactory results in solving engineering problems.

The difficulties which arise in thermal-shock design of contour-supported plates, of triple-layer plates and of plates placed on elastic foundations can be circumvented by constructing the solution in the numerical form with the use of generalized surfaces, whose ordinates are calculated on an electronic digital computer [EDC] (34BM). This approach to the solution of such a difficult problem makes it subsequently possible to obtain a formula for calculation of plates subjected to a heat flow, varying with respect to time according to any given law. By its physical nature the effect of sudden heating corresponds to a nonsteady effect and hence produces a wave process which may have a substantial influence on the stressed state of the system. The propagation of stress waves produced by the thermal shock in complex systems is a large and independent problem, for which reason its solution in the present paper will be touched upon only for the steady-state cases. For this purpose, a study is made of the reflection of thermoelastic and thermoelastoplastic waves from the end of a bar with different end supports.

The application of the general method here developed to the design of beams, frames, trusses, triple-layer plates and other elastoplastic systems makes it possible to find the distribution of bending moments and other forces in these systems.

## §2. Method of Calculation of Bar Systems

We consider a complex bar system as a system with several degrees of freedom, the mechanical model of which are concentrated coupled masses. The equation of motion of an  $i$ th mass in a system with  $n$  degrees can be written in the form [10]:

$$\delta_{i1}m_1 \frac{d^2 y_1}{dt^2} + \delta_{i2}m_2 \frac{d^2 y_2}{dt^2} + \dots + \delta_{ii}m_i \frac{d^2 y_i}{dt^2} + \dots + \delta_{in}m_n \frac{d^2 y_n}{dt^2} + y_i = \Delta_{it}, \quad (1)$$

where  $\delta_{ik}$  is the (unit) static displacement of mass  $m_i$  due to unit force applied to mass  $k$ ;  
 $\Delta_{it}$  is the temperature-induced displacement of mass  $m_i$ ;  
 $y_{id}$  is the inertia-force induced displacement of mass  $m_i$ ;

$y_i$  is the displacement of mass  $m_i$  due to temperature and inertia forces; here

$$y_i = y_{id} + \Delta_{it}. \quad (2)$$

Substituting the value of  $y_i$  from Formula (2) into Formula (1), we find

$$\begin{aligned} \delta_{i1} m_1 \frac{d^2(y_{1d} + \Delta_{1t})}{dt^2} + \delta_{i2} m_2 \frac{d^2(y_{2d} + \Delta_{2t})}{dt^2} + \dots + \delta_{in} m_n \frac{d^2(y_{nd} + \Delta_{nt})}{dt^2} + \dots + \\ + \delta_{in} m_n \frac{d^2(y_{nd} + \Delta_{nt})}{dt^2} + (y_{id} + \Delta_{it}) = \Delta_{it}. \end{aligned} \quad (3)$$

Performing elementary transformations, we find

$$\begin{aligned} \delta_{i1} m_1 \frac{d^2 y_{1d}}{dt^2} + \delta_{i2} m_2 \frac{d^2 y_{2d}}{dt^2} + \dots + \delta_{in} m_n \frac{d^2 y_{nd}}{dt^2} + y_{id} = \\ = -[\delta_{i1} m_1 \ddot{\Delta}_{1t} + \delta_{i2} m_2 \ddot{\Delta}_{2t} + \dots + \delta_{in} m_n \ddot{\Delta}_{nt}]. \end{aligned} \quad (4)$$

The right-hand side of these equations is a function of time and is evaluated by solving the thermal-conductivity problem. Quantities  $m_i \ddot{\Delta}_{it}$  can be regarded as forces, and to use for the solution of the problem methods of dynamic calculations of systems with a finite number of degrees of freedom subjected to forces  $w_i(t)$

$$w_i(t) = m_i \ddot{\Delta}_{it}. \quad (5)$$

Then, for the induced vibrations produced by the temperature, we get

$$\begin{aligned} \bar{y}_{id} = \frac{1}{\omega_i} \left\{ \int_0^t \left[ \frac{A_{i1}}{m_1} w_1(u) + \frac{A_{i2}}{m_2} w_2(u) + \dots \right] \rho_{i1} \sin \omega_i(t-u) du + \right. \\ \left. + \int_0^t \left[ \frac{A_{i1}}{m_1} w_1(u) + \frac{A_{i2}}{m_2} w_2(u) + \dots \right] \rho_{i2} \sin \omega_i(t-u) du + \dots \right\} \end{aligned} \quad (6)$$

Frequencies  $\omega_i$ , ordinates of the principal modes of vibration  $\rho_{ik}$  and the amplitudes of vibrations due to unit velocities  $A_{ik}$ , contained in Formula (6) are calculated in the ordinary manner.

The total solution of Eq. (4) is obtained by adding terms taking into account the free vibrations

$$y_{id} = \bar{y}_{id} + \Sigma (A_i \sin \omega_i t + B_i \cos \omega_i t). \quad (7)$$

The arbitrary constants  $A_i$  and  $B_i$  are determined from initial conditions.

It follows from Formula (5) that the generalized forces  $w_i(t)$  are proportional to those accelerations which are produced by temperature displacements  $\ddot{\Delta}_{it}$ , for which reason it is expedient to

analyze these accelerations for different relationships governing variations in the external temperature field.

For bar systems it is expedient to break up the temperature field into the symmetric and inverse symmetric components. For the symmetric component of the temperature field the bars of the system are heated symmetrically over their thickness and  $\Delta_{it}$  is calculated by ordinary methods of structural mechanics. For bar systems  $\Delta_{it}$  varies proportionally to the function of temperature variation  $T_{x,y,t}$ , for which reason  $\ddot{\Delta}_{it}$  will be proportional to  $\ddot{T}_{x,y,t}$ , which circumstance is of substantial importance in constructing engineering calculations.

If it is assumed that  $T_{x,y,t}$  varies linearly with time, then  $\ddot{T}_{x,y,t}=0$ , i.e.  $\ddot{\Delta}_{it}=0$  and  $w_i(t)=0$  and, according to Formula (7), it remains to study the free vibrations produced by initial conditions. This circumstance is of substantial importance in calculating systems with varying physical characteristics. If  $T_{x,y,t}$  varies parabolically with time, then forces  $w_i(t)=w_i$  become constant in time.

Finally, the third important case is obtained on sudden heating, when  $T_{x,y,t}$  is proportional to  $e^{-at}$ . Here  $w(t)$  will be proportional to  $a^2 e^{-at}$ .

Finally, when  $T_{x,y,t}$  varies according to the harmonic law  $\sin \theta t$  the generalized forces  $w_i(t)$  will vary according to the law  $\theta^2 \sin \theta t$ .

For elastic systems with several degrees of freedom, these fundamental cases have been sufficiently well studied, and hence the dynamic effect produced by sudden heating can be calculated.

Heating of an element of a system changes its elastic modulus and its coefficient of linear expansion and the bar material takes on ductile and creep properties. At first it is expedient to restrict ourselves to the study of the effect of variation in modulus  $E(T)$  and coefficient  $\alpha(T)$ . In this case quantities  $\delta_{it}(T)$  and  $\Delta_{it}(T)$  in Eq. (4) become functions of the temperature. The type of these functions depends on the material's properties and is determined experimentally. In order to make possible to use of linear approximation, the curved graphs of functions  $E(T)$  and  $\alpha(T)$  are replaced by stepped curves for values  $T_i = \left(n - \frac{1}{2}\right)\Delta t$  ( $n=1, 2, 3$ ). The stepped graph has ordinates  $E(T_i)$  corresponding to the middle of interval  $\Delta T$ .

Within the limits of the  $i$ th interval of  $\Delta t$ , the temperature variation, values of  $E_i$  and  $\alpha_i$  remain constant, for which reason Formulas (6) and (7) can be used for this time interval. Thus, the displacement of the  $i$ th mass will be calculated from the formula

$$(y_{id})_i = (y_{id})_i + \sum [(A_i)_i \sin \omega_i t + (B_i)_i \cos \omega_i t] \dots, \quad (8)$$

with the arbitrary constants  $(A_i)$  and  $(B_i)$  determined from conditions corresponding to the end of the preceding time interval. Calculation of deflections starts with the first time interval which extends from 0 to  $\Delta_1 t$ . At the end of this interval the system will have deflections and velocities which are determined from Formulas (6) and (7) with values of  $E(T) = E_1$  and  $\alpha(T) = \alpha_1$ .

Within the limits of time interval  $\Delta_1 t$  the system is considered as an elastic system with modulus  $E_1$  and with  $\alpha_1$ . Starting with time  $\Delta_1 t$  to  $t = \Delta_2 t$  the system is replaced by another, with  $E_2$  and  $\alpha_2$ . The motion of this new system is brought about by the initial displacement and the initial velocity which corresponded to the end of the first time interval  $\Delta_1 t$  and to the external temperature field which corresponds to the second time interval. Quantities  $\Delta T$  and  $\Delta t$  are interrelated by the expression

$$\Delta T = \dot{T} \Delta t. \quad (9)$$

For an external temperature field varying linearly with time  $\dot{T} = \text{const} = k$ , for which reason  $\Delta T = k \Delta t$  and  $\Delta_{tt} = 0$ .

The motion of the system depends solely on the initial conditions which vary on passing from one time interval to another, i.e., from an elastic system with one set of characteristics, corresponding to the  $i$ th time interval ( $i$ ), to another elastic system with characteristics corresponding to time interval ( $i + 1$ ). The computational volume in this solution is great and depends on the number of intervals into which the entire segment of the system's motion is broken up. In individual cases it is possible to obtain final formulas in the general form. For example, the following formula is obtained for the deflection of a beam system, corresponding to time interval  $\Delta_{nt}$ :

$$y_n(x, t) = \sum_{i=1,3,\dots} \sin \frac{i\pi x}{l} \left[ \left( \frac{2}{l\omega_{in}} \int_0^l y'_{(n-1)} \sin \frac{i\pi x}{l} dx \right) \sin \omega_{in} t + \left( \frac{2}{l} \int_0^l y_{(n-1)} \sin \frac{i\pi x}{l} dx \right) \cos \omega_{in} t + \frac{2}{l\mu\omega_{in}} \int_0^l \frac{2w_n}{\pi i} \sin \omega_{in}(t-u) du \right] \quad (10)$$

For a system with one degree of freedom, we have

$$y_n = y_0 \left( \sin \omega_1 \tau_1 \sin \omega_2 \tau_2 \sin \omega_3 \tau_3 + \frac{\omega_1}{\omega_2} \cos \omega_1 \tau_1 \cos \omega_2 \tau_2 \sin \omega_3 \tau_3 + \dots \right); \quad (11)$$

$y_0$  is the deflection at the end of the 1st interval;  $\tau_1, \tau_2, \dots, \tau_n$  are time intervals.

The method of stepped approximation makes it possible to study the motion of a system with several degrees of freedom, in which physical parameters are functions of the temperature and vary with time; here individual elements of the system may be plastic. For viscoplastic systems even with a linearly varying temperature field Eqs. (4) will be inhomogeneous and  $\Delta_{tt} \neq 0$ ; hence the solution becomes more complicated when moving from one time interval to another. The same is true with respect to systems with creep. The computational procedure for this case will be illustrated by specific examples.

### §3. Analysis of nonlinear vibrations

Nonlinear vibrations arising on sudden heating are most expediently examined for a system with two degrees of freedom. For a linearly varying temperature equations of motion (4) will be homogeneous and they can be transformed to the form

$$\left. \begin{aligned} m_1 \ddot{y}_1 + \frac{\delta_{12}}{\delta_{11}} m_2 \ddot{y}_2 + \frac{1}{\delta_{11}} y_1 &= 0; \\ \frac{\delta_{21}}{\delta_{22}} m_1 \ddot{y}_1 + m_2 \ddot{y}_2 + \frac{1}{\delta_{22}} y_2 &= 0. \end{aligned} \right\} \quad (12)$$

Quantities  $\frac{1}{\delta_{11}} = r_{11}$  and  $\frac{1}{\delta_{22}} = r_{22}$  are unit reactions which, for a system with variable physical characteristics, can be considered as functions of the displacement of masses, since the rigidity of the couplings situated between the masses will change with heating. This makes it possible to take into account in integral form the effect of variation in individual physical parameters. The variation in the restoring force of the spring, as a function of the mass displacement must be obtained experimentally or a given theoretical model must be assumed. The best possibilities with respect to this are given by representation of the restoring force in the form of polynomials, whose coefficients are chosen for the conditions of the given problem.

Transforming Eqs. (12), we get

$$\left. \begin{aligned} m_1 \ddot{y}_1 &= -\frac{1}{\delta_{22} \delta_{11} - \delta_{12}^2} [\delta_{22} y_1 - \delta_{12} y_2]; \\ m_2 \ddot{y}_2 &= -\frac{1}{\delta_{22} \delta_{11} - \delta_{12}^2} [-\delta_{21} y_1 + \delta_{11} y_2]. \end{aligned} \right\} \quad (13)$$

The expressions in brackets are represented in the form

$$\begin{aligned} \delta_{22} y_1 - \delta_{12} y_2 &= \delta'_{22} y_1 + \delta_{12} y_1 - \delta_{12} y_2 = \delta'_{22} y_1 + \delta_{12} (y_1 - y_2); \\ -\delta_{21} y_1 + \delta_{11} y_2 &= -\delta_{21} y_1 + \delta'_{11} y_2 + \delta_{11} y_2 = \delta'_{11} y_2 - \delta_{21} (y_1 - y_2), \end{aligned}$$

where

$$\delta_{22} = \delta'_{22} + \delta_{12} \text{ and } \delta_{11} = \delta'_{11} + \delta_{12}.$$

Further, using (13), we have

$$\left. \begin{aligned} m_1 \ddot{y}_1 &= -\frac{\delta'_{22}}{\delta_{22} \delta_{11} - \delta_{12}^2} y_1 - \frac{\delta_{12}}{\delta_{22} \delta_{11} - \delta_{12}^2} (y_1 - y_2); \\ m_2 \ddot{y}_2 &= -\frac{\delta'_{11}}{\delta_{22} \delta_{11} - \delta_{12}^2} y_2 + \frac{\delta_{21}}{\delta_{22} \delta_{11} - \delta_{12}^2} (y_1 - y_2). \end{aligned} \right\} \quad (14)$$

The unit displacements are calculated from the well-known expressions

$$\delta_{ik} = \sum \int \frac{M_i M_k}{EJ} ds + \sum \int \frac{N_i N_k}{EF} ds. \quad (15)$$

In the above formula the elastic modulus is a function of the temperature, but since it is contained in all the expressions of the type of Formula (15), ratio  $\delta_{ik}/\delta_{nm}$  will be constant. Formulas (14) are now represented as

$$\left. \begin{aligned} m_1 \ddot{y}_1 &= -\frac{1}{\delta_{11}} \cdot \frac{\left(1 - \frac{\delta_{12}}{\delta_{22}}\right)}{\left(1 - \frac{\delta_{12}}{\delta_{11}} \cdot \frac{\delta_{12}}{\delta_{22}}\right)} y_1 - \frac{1}{\delta_{12}} \cdot \frac{\frac{\delta_{12}}{\delta_{11}} \cdot \frac{\delta_{12}}{\delta_{22}}}{\left(1 - \frac{\delta_{12}}{\delta_{11}} \cdot \frac{\delta_{12}}{\delta_{22}}\right)} \times \\ &\quad \times (y_1 - y_2) = -r_{11} \alpha y_1 - r_{12} \gamma (y_1 - y_2); \\ m_2 \ddot{y}_2 &= -\frac{1}{\delta_{22}} \cdot \frac{\left(1 - \frac{\delta_{12}}{\delta_{11}}\right)}{\left(1 - \frac{\delta_{12}}{\delta_{11}} \cdot \frac{\delta_{12}}{\delta_{22}}\right)} y_2 + \frac{1}{\delta_{12}} \cdot \frac{\frac{\delta_{12}}{\delta_{11}} \cdot \frac{\delta_{12}}{\delta_{22}}}{\left(1 - \frac{\delta_{12}}{\delta_{11}} \cdot \frac{\delta_{12}}{\delta_{22}}\right)} \times \\ &\quad \times (y_1 - y_2) = -r_{22} \beta y_2 + r_{12} \gamma (y_1 - y_2), \end{aligned} \right\} \quad (16)$$

where  $\alpha$  and  $\beta$  are numerical coefficients, while unit reactions  $r_{11}, r_{22}$  and  $r_{12}$  should be determined experimentally for the given system.

The theoretical study may be extended to a sufficiently wide range of problems by representing these reactions in the form of the polynomials

$$r_{11} = \sum \frac{a_k}{\alpha} y_1^{k-1}; \quad r_{22} = \sum \frac{b_k}{\beta} y_2^{k-1} \text{ и } r_{12} = \sum \frac{A_k}{\gamma} (y_1 - y_2)^{k-1}. \quad (17)$$

Now we write Eqs. (16) in the form

$$\left. \begin{aligned} m_1 \ddot{y}_1 &= -\sum a_k y_1^k - \sum A_k (y_1 - y_2)^k; \\ m_2 \ddot{y}_2 &= -\sum b_k y_2^k + \sum A_k (y_1 - y_2)^k \end{aligned} \right\} \quad (18)$$

and obtain their solution according to [12, 13] using potential function  $U$ , introducing in the process the new variables

$$\xi = \sqrt{m_1} y_1 \text{ and } \eta = \sqrt{m_2} y_2; \quad (19)$$

$$U = -\frac{a_k}{k+1} \left| \frac{\xi}{\sqrt{m_1}} \right|^{k+1} - \frac{b_k}{k+1} \left| \frac{\eta}{\sqrt{m_2}} \right|^{k+1} - \frac{A_k}{k+1} \left| \frac{\xi}{\sqrt{m_1}} - \frac{\eta}{\sqrt{m_2}} \right|^{k+1}. \quad (20)$$

The equations of motion have the form

$$\ddot{\xi} = \frac{\partial U}{\partial \xi} \text{ and } \ddot{\eta} = \frac{\partial U}{\partial \eta}. \quad (21)$$

On sudden heating we have the following initial conditions

$$\xi_1(0) = \Delta_1 / \sqrt{m_1} \text{ and } \xi_2(0) = \frac{\Delta_2 t}{\sqrt{m_2}}.$$

For the skeletal curve corresponding to the second mode of vibrations [13], we write the equation

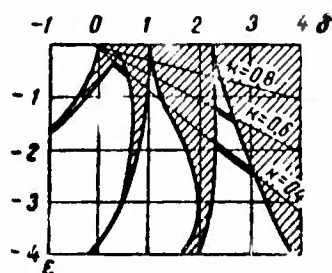


Fig. 1

$$v_{i1}^2 \left( K, \frac{\Delta_{it}}{\sqrt{m_i}} \right) = \left[ \frac{\Gamma \left( \frac{1}{k+1} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{k+1} \right)^2} \right]^2 \pi \frac{k+1}{2} m_i^{\frac{k+1}{2}} \times \times \left[ a_k + \left( 1 - \sqrt{\frac{m_1}{m_2}} \rho_{i1} \right)^k \right] \left( \frac{\Delta_{it}}{\sqrt{m_i}} \right)^{k-1}. \quad (22)$$

Here  $\Gamma$  denotes the gamma function and  $a_k = a_h/A_k$ . The skeletal curve for  $0 < k < 1$  corresponds to a reduction in rigidity brought about by heating.

Domains of unstable motion can be studied by transforming the equation of motion in an equation with variable coefficients. This was done by Rosenberg [13, 17] who has obtained instability domains in Mathieu coordinates

$$\varepsilon = 2 \frac{k-1}{k+1} \bar{\delta}. \quad (23)$$

Parameter  $\bar{\delta}_{i1}$  for each mode of vibrations is calculated from the formula

$$\bar{\delta}_{i1} = 2^{-k} v_{i1}^{-2} m_i^{-\frac{k+1}{2}} k \left[ a_k + \left( 1 - \mu^{1/2} \rho_{i1} \right)^{k-1} \right] \left( \frac{\Delta_{it}}{\sqrt{m_i}} \right)^{k-1} \frac{\Gamma(k)}{\left[ \Gamma \left( \frac{k+1}{2} \right) \right]^2}. \quad (24)$$

Each value of  $k$  has corresponding to it a straight line on the  $\varepsilon\bar{\delta}$  plane, and points corresponding to the given frequency and mode of vibrations lie on this line. When the rigidity of couplings decreases due to heating,  $k < 1$ , hence the straight lines defined by Eq. (23) will lie below the axis  $\varepsilon = 0$  (Fig. 1). The smaller  $k$ , the steeper the straight line and the greater its segment devolving upon the domain with unstable motion. When  $k$  is varied from 1 to 0.5, we get a band of changes in  $\delta$  from 0.2 to 0.6.

#### §4. Thermoelastic Vibrations of Trusses, Beams and Frames

Let us consider a statically determinate cantilevered beam truss, whose bars are suddenly, uniformly and symmetrically heated to temperature  $T_0$  degrees. To determine  $\Delta_{it}$  contained in equations of motion (1), we must construct a diagram of the displacement of the truss junctions, taking into account that each bar is elongated by  $\alpha T_0 d$ . The strained state of the truss will be somewhat unusual, since all the bars will elongate and the diagram will be a polygon similar to the outline of the truss.

The truss is considered as a system with several degrees of freedom, and it is assumed that the mass of the truss is concentrated in the joints. On sudden heating, each joint will have at

the initial time instant its initial displacement with respect to magnitude and direction. Thus, for example, the upper joints of the truss are imparted horizontal displacements, while the joints of the bottom flange of the truss will be subjected to both vertical and horizontal displacements. The displacements of each of the joints will be of different magnitudes and the total number of degrees of freedom will be 10. It can, however, be assumed that the horizontal frequencies will be appreciably higher than the vertical frequencies, and hence it is possible to separately determine the two frequencies. If we restrict ourselves to only the principal frequency of vertical vibrations of the truss, we get

$$y_{d1} = \alpha T_0 d \cos \omega_1 t; \quad \ddot{y}_{d1} = \alpha T_0 d \omega_1^2 \cos \omega_1 t = \frac{\alpha T_0 d}{m_1 \sum \frac{N_1^2}{EF} s} \cos \omega_1 t.$$

If  $EF$  is constant for all the bars, then

$$\ddot{y}_{d1} = \frac{\alpha T_0 d EF}{m_1 \sum N_1^2 s} \cos \omega_1 t \text{ and } m_1 \ddot{y}_{d1} = \frac{\alpha T_0 d EF}{\sum N_1^2 s} \cos \omega_1 t.$$

The greatest effect  $S$  produced by this force will be in bar  $BC$

$$(S_{BC})_{\max} = \frac{\alpha T_0 EF}{\sum N_1^2 s/d} \text{ and } (\sigma_{BC})_{\max} = \alpha T_0 E \frac{1}{\sum N_1^2 s/d}.$$

The stresses are proportional to the elastic modulus, linear expansion coefficient and the temperature. For example, for steel  $\alpha = 2 \cdot 10^{-5}$ ,  $E = 2 \cdot 10^6$  kg/cm<sup>2</sup>,  $\alpha E = 40$  kg/cm<sup>2</sup> and  $\sum N_1^2 s/d = 12.6$ , whence

$$(\sigma_{BC})_{\max} = \frac{40}{12.6} T_0 = 3.2 T_0.$$

The frequency of horizontal vibrations will be higher

$$y_{d1} = 2\alpha T_0 d \cos \omega_2 t; \quad \ddot{y}_{d1} = 2\alpha T_0 d \omega_2^2 \cos \omega_2 t = 2\alpha T_0 d \frac{\cos \omega_2 t}{m_1 \sum \frac{N_1^2}{EF} s};$$

$$(S_{BC})'_{\max} = \frac{2\alpha T_0 EF}{\sum N_1^2 s/d} \text{ and } (\sigma_{BC})'_{\max} = 2\alpha T_0 E \frac{1}{\sum N_1^2 s/d} = \frac{60}{4} T_0 = 20 T_0.$$

The coefficients of  $T_0$  have the dimensions kg/cm<sup>2</sup>·deg<sup>-1</sup>.

High-frequency vibrations produce high stresses in the bars of the truss. For a given truss the stresses are proportional to the temperature.

If the elastic modulus is temperature dependent, then the frequency of the system will change during the vibrations. Here the magnitude of stresses depends on the rate of change in the system's frequency. The effect of this factor can be estimated by means of Formula (11), from which it follows that the expression in parentheses depends appreciably on the size of time intervals  $\tau_i$  during which the elastic properties of the system change. Figure 2 shows graphs of variation in the maximum stress of bar  $BC$  as a function of the ratio of frequency  $\omega_k$  for a variable elastic

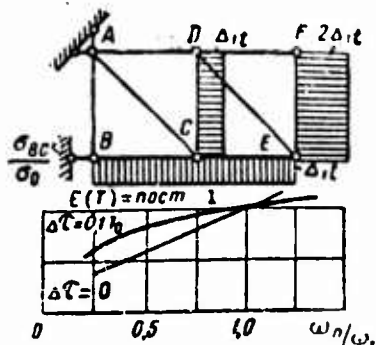


Fig. 2. 1) Constant.

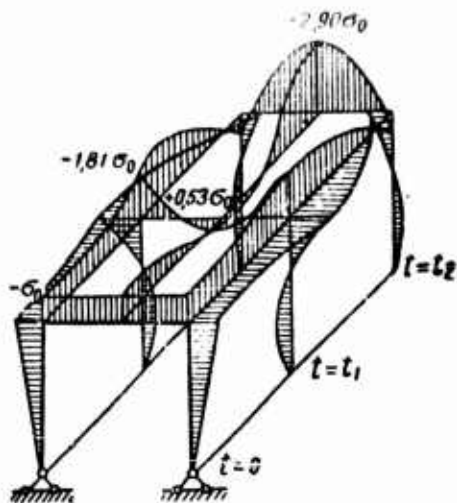


Fig. 3

modulus of the truss material, to frequency  $\omega_0$  for a truss with constant modulus. The graph shows that the slower the properties of the system change, the less are the stresses in the bar reduced.

The greatest stresses are obtained in a system with constant characteristics, while the smallest prevail in a system in which the modulus changes instantaneously.

The solution in a statically indeterminate truss is complicated by the need to calculate  $\Delta_{it}$  which must be found in a statically indeterminate system. For an elementary equilateral triangular truss  $\Delta_{it} = 1.422 a d T_0$ ; here on static application of temperature the force in bar  $CB$  is  $S = +0.41 a T_0 E F$ . We now take into account the effect of a gradual increase in temperature linearly to  $T_0$  during time  $\tau$ , which is less than one quarter of a period of the fundamental mode of vibration of the truss. From Eqs. (6) and (7) we also get formulas for the vertical deflection due to inertia forces

$$y_{id} = \Delta_{it} \frac{\sin \frac{\omega_1 \tau}{2}}{\omega_1 \tau / 2} \cos \left( \omega_1 t - \frac{\omega_1 \tau}{2} \right) \text{ and } (y_{id})_{\max} = \Delta_{it} \frac{\sin \frac{\omega_1 \tau}{2}}{\omega_1 \tau / 2}.$$

The total deflection will be obtained by adding to the dynamic deflection the deflection produced by static application of temperature; hence

$$y_{\max} = \Delta_{it} \left( 1 + \frac{\sin \frac{\omega_1 \tau}{2}}{\omega_1 \tau / 2} \right).$$

On instantaneous application of temperature, i.e., for  $\tau = 0$ ,

$$(y_{\max})_{\tau=0} = 2\Delta_{it} = 2 \cdot 1.422 a d T_0 = 2.844 a d T_0.$$

On sudden heating, the deflection obtained from static application of temperature is thus doubled. The dynamic part of the deflection produces large forces due to the fact that the unit reactions corresponding to different bars are calculated for a statically indeterminate system, which is a more rigid system. We calculate

$$(S_{AB})_{dyn} = (S_{AB})_{cr} \frac{1}{\delta_{11}} y_{1d} = 0,6 \Delta_{1f} \frac{\sin \frac{\omega_1 \tau}{2}}{\omega_1 \tau / 2} \cdot \frac{EF}{0,568d}.$$

The total maximum force is

$$(S_{AB})_{max} = (S_{AB})_{dyn} + (S_{AB})_{cr} = 0,6 \frac{EF}{0,568d} 1,422 \alpha d T_0 + \\ + 0,41 \alpha T_0 EF = 1,91 \alpha T_0 EF.$$

As compared with static application of temperature, force  $S_{AB}$  increased by  $1.91/0.41 = 4.6$ . This interesting property of statically indeterminate systems must be remembered when designing them for strength. It cannot be claimed that the magnification factor on sudden heating is two. For a force this coefficient will be more than two, as can be seen from the above example.

Let us consider a  $\Pi$ -shaped frame (Fig. 3) whose crossbar is suddenly and uniformly heated to  $T_0$ ; this system is peculiar by the fact that uniform heating in this case produces bending of the crossbar at the supports of the frame. In this case, the midspan vertical deflection on static application of  $T_0$  is  $\Delta_{1f} = \frac{3}{40} \alpha T_0 l$ , while

the dynamic deflection is  $y_{1d} = \frac{3}{40} \alpha T_0 l \frac{\sin \frac{\omega_1 \tau}{2}}{\omega_1 \tau / 2}$ . We now compare the bending moments produced at midspan by: a) static application of temperature

$$M_{cr} = \frac{3}{5} \alpha T_0 \frac{EJ}{l} = 0,6 \alpha T_0 \frac{EJ}{l};$$

b) inertia forces produced on sudden heating

$$(M_{dyn})_{np} = y_{1d} \mu_{np} = \frac{3}{40} \alpha T_0 l \frac{\sin \frac{\omega_1 \tau}{2}}{\omega_1 \tau / 2} 15,3 \frac{EJ}{l^3} = 1,13 \alpha T_0 \frac{EJ}{l} \cdot \frac{\sin \frac{\omega_1 \tau}{2}}{\omega_1 \tau / 2},$$

where  $\mu_{np} = 15,3 \frac{EJ}{l^3}$  is the midspan bending moment produced by unit crossbar displacement. The maximum midspan moment, taking inertia forces into account is  $1,73/0,6 \approx 2.9$  greater than that obtained on static temperature application.

We get the second equation for the support moment, since the moment at the support due to unit midspan displacement is

$$\mu_{on} = \frac{72}{11} \cdot \frac{EJ}{l^3}, \quad \text{hence} \quad (M_{\text{дин}})_{on} = \frac{3}{40} \alpha T_0 l^2 \frac{72}{11} \cdot \frac{EJ}{l^3} = 0.9 T_0 \frac{EJ}{l},$$

$$M_{\text{дин}} = M_{\text{ст.он}} + (M_{\text{дин}})_{on} = 1.09 T_0 \frac{EJ}{l} \quad \text{and the ratio} \quad \frac{1.09}{0.6} = 1.81.$$

For the study of variations in the moment diagram with respect to time it is expedient to construct an influence surface, as this is done in Fig. 3. Each ordinate of the diagram consists of two components, one of which is the bending moment produced by static application of the temperature field. This moment does not depend on time. The second part, which depends on inertia forces, is a function of the coordinates of cross section, as well as of time. It can be concluded by analyzing this generalized diagram that at the beginning it is the support moments which reach the maximum, with the maximum in the span obtained much later.

This circumstance is of great significance for determining the position of plastic hinges when the system goes past the elastic limit. The first two plastic hinges are formed on the supports of the crossbar, after which the properties of the system change, since each bar of the frame will represent a beam hinged at the ends. The moment distribution in the system changes and the mid-span bending moment will be equal to the limiting value.

## §5. Thermoelastoplastic Vibrations of Plates

In the preceding section we have considered vibrations which arise on sudden uniform heating. To study the nonsymmetric case of heating we now consider a triple-layer plate consisting of two plates separated by an elastic filler. The particular cases which follow from this more general solution are plates on an elastic foundation and plates hinged along the contour.

If heat flux  $q$  is applied suddenly to the upper plate, then the vibrations of this triple-layer plate can be represented by the following system of equations of motion

$$D_1 \nabla^4 w_1 - N_1 + \mu \frac{\partial^2 w_1}{\partial t^2} = 0; \quad D_2 \nabla^4 w_2 + N_1 + \Delta N_1 + \mu_2 \frac{\partial^2 w_2}{\partial t^2} =$$

$$= - \frac{1}{1-\nu} \nabla^2 M_T. \quad (25)$$

Here  $D_1, D_2$  are the cylindrical rigidities of the lower and upper plate;  $w_1$  and  $w_2$  are the deflections of the lower and upper plate;  $N_1$  is the reaction in the elastic couplings;  $\Delta N_1$  is the inertia force of the elastic couplings;  $\nu$  is Poisson's ratio;  $M_T$  is the bending moment produced in the upper plate by static application of temperature

$$M_T = \alpha E \int_{-h/2}^{+h/2} T z dz; \quad (26)$$

$T$  is the temperature distribution function. According to [11]

$$M_T = \alpha E \frac{qh^3}{24\lambda} \left( 1 - \frac{96}{\pi^4} \sum_{n=1,3,\dots} \frac{1}{n^4} e^{-n^2 \pi^2 h^2 / 4\lambda^2} \right). \quad (27)$$

where  $\alpha$  is the linear expansion coefficient of the plate's material;  $E$  is the elastic modulus;  $h$  is the thickness of the upper plate;  $k$  is the coefficient of thermal diffusivity;  $\lambda$  is the coefficient of thermal conductivity. Forces  $N_1$  and  $\Delta N_1$  are expressed in terms of deflections  $w_1$  and  $w_2$  by the formulas

$$N_1 = \left( \frac{w_2 - w_1}{h_0} \right) D_3 - \frac{\mu_0 h_0}{3} \cdot \frac{\partial^2 w_1}{\partial t^2} - \frac{\mu_0 h_0}{6} \cdot \frac{\partial^2 w_2}{\partial t^2} \text{ and} \\ \Delta N_1 = 0,5 h_0 \mu_0 \left( \frac{\partial^2 w_1}{\partial t^2} + \frac{\partial^2 w_2}{\partial t^2} \right); \quad (28)$$

$h_0$  is the thickness of the elastic layer;  $\mu_0$  is the mass of unit volume of the elastic layer;  $D_3$  is the stiffness in compression of the elastic layer. Substituting  $N_1$  and  $\Delta N_1$  into Expression (25), we get

$$\left. \begin{aligned} D_1 \nabla^4 w_1 + \left( \mu_1 + \frac{h_0 \mu_0}{3} \right) \frac{\partial^2 w_1}{\partial t^2} + \frac{\mu_0 h_0}{6} \cdot \frac{\partial^2 w_2}{\partial t^2} - (w_2 - w_1) \frac{D_3}{h_0} &= 0; \\ D_2 \nabla^4 w_2 + \left( \mu_2 + \frac{h_0 \mu_0}{2} \right) \frac{\partial^2 w_2}{\partial t^2} + \frac{\mu_0 h_0}{6} \cdot \frac{\partial^2 w_1}{\partial t^2} + (w_2 - w_1) \frac{D_3}{h_0} &= \\ &= - \frac{1}{1 - \nu} \nabla^2 M_T. \end{aligned} \right\} \quad (29)$$

We first consider the case of symmetrical heating, when an amount of heat  $q$  is suddenly and uniformly applied to the entire areas of the lower and upper plates. Then  $w_2 = -w_1 = w$  and from Eq. (29) we get one equation for determining  $w$

$$\frac{1}{2} (D_1 + D_2) \nabla^4 w + \frac{1}{2} \left( \mu_1 + \mu_2 + \frac{h_0 \mu_0}{3} \right) \frac{\partial^2 w}{\partial t^2} + w \frac{2D_3}{h_0} = \\ = - \frac{1}{2} \cdot \frac{1}{1 - \nu} \nabla^2 M_T.$$

If, on the other hand,  $D_1 = D_2 = D$  and  $\mu_1 = \mu_2 = \mu$ , we get

$$D \nabla^4 w + \left( \mu + \frac{h_0 \mu_0}{6} \right) \frac{\partial^2 w}{\partial t^2} + w \frac{2D_3}{h_0} = - \frac{1}{2} \cdot \frac{1}{1 - \nu} \nabla^2 M_T. \quad (30)$$

For the inverse symmetrical case of sudden heating we get, by subtracting the second of Eqs. (28) from the first and setting  $w_1 = w_2$

$$D \nabla^4 w + \left( \mu + \frac{h_0 \mu_0}{2} \right) \frac{\partial^2 w}{\partial t^2} = - \frac{1}{2(1 - \nu)} \nabla^2 M_T. \quad (31)$$

We present the solution of these two equations in the form considered in the second section

$$w_i = w_{si} + w_{di} \quad (i = 1, 2); \quad (32)$$

$w_{si}$  is the deflection due to static application of temperature; this deflection corresponds to Eqs. (30) and (31) without consideration of the inertia terms; hence we get for  $w_{di}$ :

a) on symmetric heating

$$D\nabla^4 w_{d1} + \left(\mu + \frac{\mu_0 h_0}{6}\right) \ddot{w}_{d1} + \frac{2D_3}{h_0} w_{d1} = -\left(\mu + \frac{\mu_0 h_0}{6}\right) \ddot{w}_{s1} \quad (33)$$

b) on inverse symmetric heating

$$D\nabla^4 w_{d2} + \left(\mu + \frac{\mu_0 h_0}{2}\right) \ddot{w}_{d2} = \left(\mu + \frac{\mu_0 h_0}{2}\right) \ddot{w}_{s2} \quad (34)$$

We seek the solution of these equations as the series

$$w_{si} = \sum \sum (K_{nm})_i \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \quad (i = 1, 2); \quad (35)$$

$$w_{di} = \sum \sum [q_{nm}(t)]_i \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \quad (i = 1, 2). \quad (36)$$

For the coefficients of the expansion [14] we get

$$(K_{nm})_1 = - \frac{16M_T (-1)^{\frac{m+n}{2}} \left[ \left( \frac{n\pi}{a} \right)^2 + \left( \frac{m\pi}{b} \right)^2 \right]}{2(1-\nu) \pi^2 D \left\{ nm \left[ \left( \frac{n\pi}{a} \right)^2 + \left( \frac{m\pi}{b} \right)^2 \right]^2 + \frac{2D_3}{Dh_0} \right\}}; \quad (37)$$

$$(K_{nm})_2 = - \frac{16M_T (-1)^{\frac{m+n}{2}}}{2(1-\nu) D \pi^2 nm \left[ \left( \frac{n\pi}{a} \right)^2 + \left( \frac{m\pi}{b} \right)^2 \right]}. \quad (38)$$

Functions  $[q_{nm}(t)]_i$  are solutions of the equation

$$(\ddot{q}_{nm})_i + (\omega_{nm}^2)_i (q_{nm})_i = -(\bar{K}_{nm})_i \ddot{M}_T \quad (39)$$

The solution of this equation for function  $M_T$  obtained from Formula (27) was obtained in [15] and has the form

$$(q_{nm})_i = \frac{q}{2\lambda} (\bar{K}_{nm})_i \left[ \frac{12\beta}{\pi^2 (\omega_{nm})_i} \sin (\omega_{nm})_i t - \frac{96\beta^3}{\pi^4} \sum_{j=1,3,5} \frac{1}{j^4 \beta^2 + (\omega_{nm})_i} \times \right. \\ \left. \times \left( \cos (\omega_{nm})_i t + \frac{(\omega_{nm})_i}{\beta} \cdot \frac{1}{j^3} \sin (\omega_{nm})_i t - e^{-\beta t} \right) \right]. \quad (40)$$

The symbol  $\beta$  denotes the quantity

$$\frac{\pi k}{h^3}; \quad \left( \beta = \frac{\pi k}{h^3} \right). \quad (41)$$

Formula (40) also contains frequencies which are different for symmetric and inverse symmetric modes:

$$(\omega_{nm}^2)_1 = \left\{ \left[ \left( \frac{n\pi}{a} \right)^2 + \left( \frac{m\pi}{b} \right)^2 \right] + \frac{2D_3}{h_0 D} \right\} \frac{D}{\left( \mu + \frac{1}{6} \mu_0 h_0 \right)}; \quad (42)$$

$$(\omega_{nm}^2)_2 = \pi^4 \left( \frac{n^2}{a^2} + \frac{m^2}{b^2} \right) \frac{D}{\left( \mu + \frac{\mu_0 h_0}{2} \right)}. \quad (43)$$

The deflections are calculated from Formulas (35) and (36) after substituting into them quantities defined by Expressions

(37), (38), (40), (42) and (43) and is a highly work-consuming problem, which can actually be solved only by using an electronic digital computer.

The influence surfaces of symmetric modes were constructed on the "Ural-2" EDC for the following starting data:  $\alpha = 12 \cdot 10^{-4}$ ;  $q = 100$  kcal;  $\nu = 0,17$ ;  $\rho = 2400$  g/cm<sup>3</sup>;  $c = 0,21$  kcal/kg°C;  $\lambda = 0,22 \cdot 10^{-3}$  kcal/m·s;  $\mu = 244,65$  kg·s<sup>2</sup>/m<sup>4</sup>;  $E = 21 \cdot 10^8$  kg/m<sup>2</sup>;  $a = 5$  m;  $b = 4$  m;  $h = 0,23$  m, for three ratios between the rigidity of the plates and elasticity of the couplings  $\eta = \frac{2D_s}{h_0 D} = 0,1; 1; 10$ .

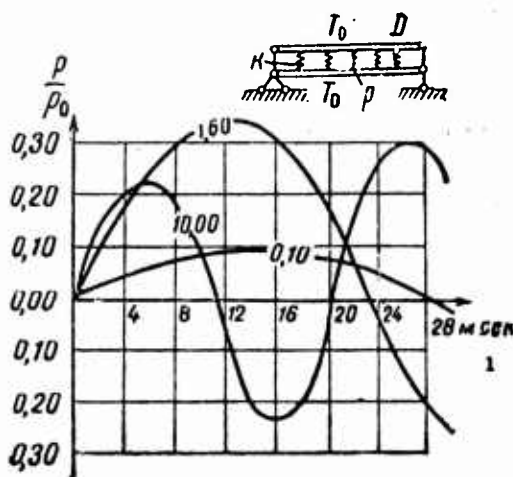


Fig. 4. 1) ms.

Graphs of variations in reactions of midspan couplings are shown in Fig. 4. It follows from them that increasing the coupling rigidity first increases the reactions in the latter, but then this process is slowed down.

## §6. Thermal Shock Calculations

The calculations of the thermal shock  $S_t = q\tau$ , where  $\tau$  is the time during which a heat flux with strength  $q$  acts upon a system in the elastic range, can be constructed by superposition of two solutions. First we shall assume that at  $t = 0$  a heat flux of strength  $q$ , uniformly distributed over the entire area, is applied to the surface of the plate. Then, after a time  $\tau$ , the same heat flux, but of opposite sign, is applied. Then the motion of the plate will be brought about by the thermal shock  $S_t = q\tau$ .

The formula for the deflection will then be

$$w_{dt} = \sum \sum \{ [q_{nm}(t)]_t - [q_{nm}(t - \tau)]_t \} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}. \quad (44)$$

We evaluate the expression in the braces using Formula (40), which yields

$$\begin{aligned}
w_{dl} = & \frac{S_l}{2\lambda} \cdot \frac{96}{\pi^4} \sum_{n=1,3,\dots} \sum_{m=1,3,\dots} (K_{nm})_l \left[ \frac{\sin \frac{(\omega_{nm})_l \tau}{2}}{(\omega_{nm})_l \tau/2} \left\{ \frac{\pi^2}{8} - \frac{\beta}{(\omega_{nm})_l} \right. \right. \\
& - \sum_{j=1,3,\dots} \frac{1}{j^4 + \left( \frac{(\omega_{nm})_l}{\beta^2} \right)^2} \sin \left( (\omega_{nm})_l t + \frac{(\omega_{nm})_l}{\beta} \cdot \frac{1}{j^2} \right) \cos \left[ (\omega_{nm})_l - \frac{((\omega_{nm})_l \tau)}{\beta^2} \right] - \\
& - \sum_{j=1,3,\dots} \left( \frac{1}{j^4 + \frac{(\omega_{nm})_l^2}{\beta^2}} \sin \left( (\omega_{nm})_l t + \frac{(\omega_{nm})_l \tau}{2} \right) \right) + \\
& + \sum_{j=1,3,\dots} \frac{(e^{-j^2 \beta^2 t})(1 - e^{j^2 \beta^2 \tau})}{\tau (\omega_{nm})_l} \left[ \frac{1}{j^4 + \frac{(\omega_{nm})_l^2}{\beta^2}} + \frac{1}{j^4} \right] \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}. \quad (45)
\end{aligned}$$

If we restrict ourselves only to consideration of the fundamental mode of vibrations and make some simplifications, then we will find the following formulas, defining the maximum deflection and maximum moment for a single-layer beam plate

$$y_{max} = \frac{6}{\pi^2} \cdot \frac{\alpha}{c\gamma} \left( \frac{l}{h} \right)^2 S_l \frac{\sin \frac{\omega \tau}{2}}{\frac{\omega \tau}{2}}; \quad M_{max} = \frac{1}{2} \cdot \frac{Eah}{c\gamma} S_l \theta. \quad (46)$$

The graph of coefficient  $\theta$  for different pulse shapes is shown in Fig. 5.

The system will go past the elastic limit with the attendant formation of a plastic hinge after the heat pulse will stop acting and when the system will perform free vibrations, and this makes it possible to find that time instant  $t_0$  for which  $M_{max}$  will be equal to  $M_{pr}$ , i.e., the limiting bending moment corresponding to the formation of the plastic hinge.

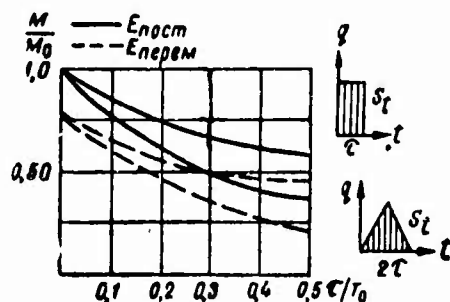


Fig. 5

At this time instant the deflections of the system can be calculated from Formula (45). If we set  $t=t_0$ , then we get the deflection  $(w_{dl})_{t=t_0} = w_{i0}$  and velocity  $(\dot{w}_{dl})_{t=t_0} = \dot{w}_{i0}$ .

Using the method presented in §2, we should now consider the motion of the new system with the rigidity reduced due to formation of the plastic hinge. This new system is set into motion with initial conditions  $(w_{dl})_{t=0} = w_{i0}$  and  $(\dot{w}_{dl})_{t=0} = \dot{w}_{i0}$  and is described by the equation

$$\begin{aligned}
\ddot{w}_{dl} = & \sum \sin \frac{i\pi x}{l} \left\{ \left[ \frac{2}{l\omega_{in}} \int_0^l \frac{\dot{w}_{i,n-1}}{n-1} \sin \frac{i\pi x}{l} dx \right] \sin \omega_{in} t + \right. \\
& + \left. \left( \frac{2}{l} \int_0^l w_{i,n-1} \sin \frac{i\pi x}{l} dx \right) \cos \omega_{in} t \right\}. \quad (47)
\end{aligned}$$

Now the deflection increases at a much higher rate, and this can be shown by comparing the graphs of midspan deflection for an elastic and elastoplastic beam plate.

## §7. The Wave Effect

The propagation of thermoelastoplastic waves in a semi-infinite bar was considered by this author in [14], where it was established that as a result of sudden heating of one end of a bar, first a longitudinal elastic stress wave is formed, which can become a plastic wave while moving along the bar. It is now expedient to estimate the effect of the interrelationship between equation of motion (48) and the thermoelasticity equations (49). According to [16] we have

$$\frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial T}{\partial x}; \quad (48)$$

$$\rho c_1 [1 + 2\varepsilon(1 - \nu)] \frac{\partial T}{\partial t} + \alpha T_0 E \frac{\partial^2 u}{\partial x \partial t} = \lambda \frac{\partial^2 T}{\partial x^2}. \quad (49)$$

We solve these equations by the method given in §2:

$$u = u_d + u_s, \quad (50)$$

where  $u$  is the total displacement,  $u_d$  is the displacement due to inertia forces,  $u_s$  is the displacement due to static application of temperature.

For  $u_s$  Eq. (48) simplifies to the form

$$\frac{\partial^2 u_s}{\partial x^2} - \alpha \frac{\partial T}{\partial x} = 0,$$

upon integrating which with respect to  $x$ , we get  $\frac{\partial u_s}{\partial x} = \alpha T$ . Differentiating now this equation with respect to  $t$  and substituting into (49), we get the classical equation of thermal conductivity  $\frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2}$ , i.e., for statically applied temperature field Eqs. (48) and (49) become separated and  $u_s$  can be obtained from ordinary solutions.

If, however, we drop the right-hand side of Eq. (49), then this equation can be integrated with respect to  $t$  and then differentiated with respect to  $x$ , and then the value of  $\frac{\partial T}{\partial x}$  can be substituted into Eq. (48). For the dynamic part of the deflection we get

$$\frac{1}{(c')^2} \cdot \frac{\partial^2 u_d}{\partial t^2} = \frac{\partial^2 u_d}{\partial x^2}, \quad (51)$$

where

$$(c')^2 = \frac{E_s}{\rho} \quad \text{or} \quad E_s = E_f \left[ 1 + \frac{e(1-v)(1-2v)}{(1+v)[1+2e(1-v)]} \right] \quad (52)$$

$$(c')^2 = \left[ 1 + \frac{e(1-v)(1-2v)}{(1+v)[1+2e(1-v)]} \right] c^2, \quad e = \frac{\alpha^2 T_0}{\rho^2 c_e k_f^2 v_p^2},$$

$$k_f = 3(1-2v)E; \quad (53)$$

$\sigma_e$  is the specific heat,  $T_0$  is the temperature of the stressed state

$$v_p^2 = \frac{E}{\rho} \cdot \frac{(1-v)}{(1+v)(1-2v)}.$$

We now consider the case when

$$T(x, t) = at e^{-bx}. \quad (54)$$

The solution of Eq. (51) now has the form

$$u_d = A e^{-b(x-c't)} + B e^{-b(x+c't)}.$$

Taking into account the fact that when  $t=0$ ,  $u_d=0$ , we get  $A=-B$  and

$$u_d = A (e^{-b(x-c't)} - e^{-b(x+c't)}). \quad (55)$$

Using the second initial condition for  $t=0$  and  $\frac{\partial u}{\partial t}=0$ , we find

$$A = + \frac{\alpha a}{2b^2 c'}. \quad (56)$$

For stress waves propagating through the bar we get the formulas

$$\sigma = - \frac{E \alpha a}{b c'} \begin{cases} e^{-bc't} \operatorname{sh}(bx) & x < c't; \\ e^{-bx} \operatorname{sh}(bc't) & x > c't. \end{cases} \quad (57)$$

The maximum stress is obtained for  $x=c't$

$$\sigma_{\max} = -0.5 \frac{E \alpha a}{b c'} (1 - e^{-2bc't}). \quad (58)$$

Let us compare this result with that obtained previously for separated Eqs. (48) and (49), for which we calculate the maximum stress ratio for these two cases

$$\sigma_{\max} / \sigma_{\max}^0 = \frac{c}{c'} \cdot \frac{(1 - e^{-2bc't})}{(1 - e^{-2bct})}. \quad (59)$$

Figure 6 shows the surface of propagation of thermoelastic waves along the bar. The solid line corresponds to the case of unrelated elasticity and thermal conductivity equations, while the dashed line was constructed upon consideration of the interrelationship between Eqs. (48) and (49). The maximum stress with consideration of this interrelationship is found to be lower, as is shown by Formulas (59). In the limit, as  $bct = \infty$ , we get

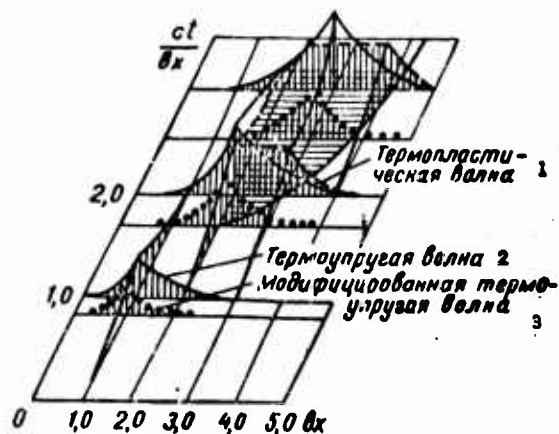


Fig. 6. 1) Thermoplastic wave; 2) thermoelastic wave; 3) modified thermoelastic wave.

$$\sigma_{\max}/\sigma_{\max}^0 = \frac{c}{c'} \sqrt{\left[1 + \frac{\epsilon(1-\nu)(1-2\nu)}{(1+\nu)[1-2\epsilon(1-\nu)]}\right]} \approx \sqrt{\left[\frac{1}{1+0.3\epsilon}\right]}. \quad (60)$$

Thus, for example, when  $\epsilon=0.5$ — $\sigma_{\max}/\sigma_{\max}^0 \approx 0.91$ , i.e., the stresses are lower by 10%.

A plastic wave will form when  $\sigma > \sigma_0$ . The stresses for this case are shown in Fig. 6. The construction was performed by the method of characteristics.

## §8. Discussion of Results

The above general method of calculating bar systems and plates for the effect of sudden heating makes it possible to determine dynamic forces in elements of these systems, taking into account variations in their physical parameters brought about by temperature. Approximate formulas were obtained for bending moments and other forces calculated for thermal shock.

Stresses due to consideration of inertia forces arise on sudden heating of statically determined systems. Formulas were obtained for calculating the stress in a statically determinate truss suddenly subjected to a temperature field.

In statically indeterminate systems, displacements on sudden heating cannot be more than two-fold greater than the displacements produced by static temperature application. But the bending moments, reactions and other forces are here increased more than two-fold. For a  $\Pi$ -shaped frame, for example,  $M_{\text{din}}$  will be 2.9-fold greater than  $M_{\text{st}}$ , due to which an unusual sequence is obtained in the formation of plastic hinges upon going past the elastic limit. A study of the effect of changes in the physical constants of a system which are produced by heating points to the need of solving nonlinear equations of motion. It is shown that the nonlinear equations which describe the thermoelastic vibrations may, as a

result of special transformations, be reduced to the ordinary form and that it is possible to construct domains of unstable motion which arise in this case. The more general problem of thermoelastic vibrations of a triple-layer plate was solved by breaking up the vibrations into symmetric and inverse symmetric modes which, unlike the ordinary problem, arise even on uniform heating of one of the plate surfaces. Two groups of vibratory modes and their corresponding frequencies were found. The solution was worked out on the "Ural-2" EDC. As a particular case, formulas for a plate on an elastic foundation were established from the obtained general solutions. The dynamic reaction of the elastic foundation are found to be greatest for a clearly defined ratio between the plate's and elastic foundation's rigidities. Uncoupling may take place for this inconvenient ratio, and the system's motion may become unstable.

The propagation of thermoelastic and thermoplastic waves was studied by taking into account the interrelationship existing between the elasticity and thermal conductivity equations, which also affects the changes in the wave-motion parameters. The velocity of the modified stress wave becomes higher, but the maximum stresses are here smaller. As a result of this, an appreciable change takes place in the stress distribution along the bar. Thus, in engineering calculations one must take into account the effect of mechanical and thermodynamic processes, which are brought about on sudden heating of bar systems and plates.

The quantities obtained on the basis of these approximate theoretical studies were compared with experimental data and the agreement was found satisfactory, for which reason the formulas can be used for engineering calculations within established limits.

Alongside with this it should be noted that the topic of this study is quite wide, and although the number of published studies of these problems increases very rapidly, this topic has by far not been exhausted. It would be more correct to assume that only a beginning has been made and that major problems are still to be solved.

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#### Footnote

- 101 <sup>1</sup>Read in January 1964 to the IInd All-Union Conference on Theoretical and Applied Mechanics.

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#### Transliterated Symbols

- 111 дин = din = dinamicheskii = dynamic  
111 ст = st = staticheskii = static  
111 т = t = temperaturnyy = temperature  
111 пр = pr = prolet = span  
112 оп = op = opora = support  
112 ст.оп = st.op = staticheski opredelimyy = statically  
determinate  
116 пост = post = postoyannyy = constant  
116 перем = perem = peremennyy = variable

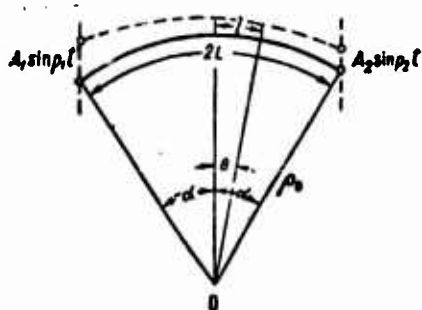
# VIBRATIONS OF VERY SHALLOW CIRCULAR ARCHES ON VIBRATING SUPPORTS

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This article presents the solution of the problem of vibrations of very shallow circular arches of constant rigidity on the assumption that the hinged ends of the arch perform vertical harmonic vibrations in general with different amplitudes and frequencies (Figure).

Solutions of this kind of problem, pertaining to the study of vibrations of straight bars are given in [1-8].



Figure

The author of [9], by transforming the known Kirchhoff-Love equations, describing the motion of a thin curved bar of any shape, has obtained the equation of motion of a bar with a circular axis of small curvature (shallow circular arc). This fourth-order nonlinear equation can be represented in dimensionless coordinates in the form

$$\frac{\partial^4 \xi}{\partial \theta^4} + \lambda^2 \xi \frac{\partial^2 \xi}{\partial \theta^2} + \lambda^2 \xi = -\lambda^2 \gamma \frac{\partial^2 \xi}{\partial t^2}, \quad (1)$$

where

$$\xi(0, t) = \frac{1}{\rho_0} u(0, t); \quad \lambda = \frac{\rho_0}{i}; \quad \gamma = \frac{\rho_0^2 \mu}{E}.$$

Here  $u(0, t)$  is the radial translation of any point on the bar's axis;  $\theta$  is a variable angle (Figure);  $\rho_0$  is the initial radius of curvature;  $i$  is the radius of gyration of the cross section;  $\mu$  is the mass per unit volume;  $E$  is the elastic modulus in tension.

An exact solution of the problem can be obtained easily if it is assumed that the translations of the points on the bar's axis are so small that Eq. (1) can be linearized. Dropping in this equation the term containing the product  $\xi \frac{\partial^2 \xi}{\partial \theta^2}$ , we get

$$\frac{\partial^4 \xi}{\partial \theta^4} + \lambda^2 \xi = -\lambda^2 \gamma \frac{\partial^2 \xi}{\partial t^2}. \quad (2)$$

We restrict ourselves to the consideration of the case when the hinged ends of the arch perform simple harmonic vibrations in the vertical direction. By virtue of the fact that the arch is very shallow, we can consider angle  $\alpha$  to be sufficiently small, which makes it possible to write the boundary conditions in the form

$$u(+\alpha, t) = A_2 \sin p_2 t; \quad u(-\alpha, t) = A_1 \sin p_1 t; \quad \left( \frac{\partial^2 u}{\partial \theta^2} \right)_{\theta=\pm\alpha} = 0 \quad (3)$$

or

$$\xi(+\alpha, t) = \frac{A_2}{\rho_0} \sin p_2 t; \quad \xi(-\alpha, t) = \frac{A_1}{\rho_0} \sin p_1 t; \quad \left( \frac{\partial^2 \xi}{\partial \theta^2} \right)_{\theta=\pm\alpha} = 0. \quad (4)$$

We seek the solution of Eq. (2) in the form

$$\xi(\theta, t) = f_1(\theta) \sin p_1 t + f_2(\theta) \sin p_2 t, \quad (5)$$

where  $f_1(\theta)$  and  $f_2(\theta)$  are functions to be determined. Substituting Eq. (5) into Eq. (2), we get

$$[f_1^{IV} + f_1 \lambda^2 (1 - \gamma p_1^2)] \sin p_1 t + [f_2^{IV} + f_2 \lambda^2 (1 - \gamma p_2^2)] \sin p_2 t = 0.$$

Subjecting then Expression (5) to boundary conditions (4), we get

$$f_1(+\alpha) \sin p_1 t + f_2(+\alpha) \sin p_2 t = \frac{A_2}{\rho_0} \sin p_2 t;$$

$$f_1(-\alpha) \sin p_1 t + f_2(-\alpha) \sin p_2 t = \frac{A_1}{\rho_0} \sin p_1 t;$$

$$\left( \frac{\partial^2 f_1}{\partial \theta^2} \right)_{\theta=+\alpha} \sin p_1 t + \left( \frac{\partial^2 f_2}{\partial \theta^2} \right)_{\theta=+\alpha} \sin p_2 t = 0;$$

$$\left( \frac{\partial^2 f_1}{\partial \theta^2} \right)_{\theta=-\alpha} \sin p_1 t + \left( \frac{\partial^2 f_2}{\partial \theta^2} \right)_{\theta=-\alpha} \sin p_2 t = 0.$$

By virtue of the fact that the support hinge movements are independent of one another, we shall arrive for determining functions  $f_1(\theta)$  and  $f_2(\theta)$  at the following differential equations and corresponding boundary conditions

$$f_1^{IV}(\theta) + f_1(\theta) \lambda^2 (1 - \gamma p_1^2) = 0, \quad (6)$$

$$\left. \begin{aligned} f_1(+\alpha) = 0; \quad f_1(-\alpha) = \frac{A_1}{\rho_0}; \quad \left( \frac{\partial^2 f_1}{\partial \theta^2} \right)_{\theta=\pm\alpha} = 0; \end{aligned} \right\} \quad (7)$$

$$f_2^{IV}(\theta) + f_2(\theta) \lambda^2 (1 - \gamma p_2^2) = 0, \quad (8)$$

$$\left. \begin{aligned} f_2(+\alpha) = \frac{A_2}{\rho_0}; \quad f_2(-\alpha) = 0; \quad \left( \frac{\partial^2 f_2}{\partial \theta^2} \right)_{\theta=\pm\alpha} = 0. \end{aligned} \right\} \quad (9)$$

Equations (6) and (8) will be integrated on the following two assumptions.

1.  $1 - \gamma p_1^2 < 0$  and  $1 - \gamma p_2^2 < 0$ , which is equivalent to the inequalities

$$p_1 > \sqrt{\frac{E}{\mu \rho_0^2}}; \quad p_2 > \sqrt{\frac{E}{\mu \rho_0^2}}. \quad (10)$$

2.  $1 - \gamma p_1^2 > 0$  and  $1 - \gamma p_2^2 > 0$ , or, which is the same thing,

$$p_1 < \sqrt{\frac{E}{\mu \rho_0^2}}; \quad p_2 < \sqrt{\frac{E}{\mu \rho_0^2}}. \quad (11)$$

The first case corresponds to vibrations of the arch for which the frequencies  $p_1$  and  $p_2$  of the vibrations of its supports is higher than the frequency of tension-compression vibrations of the arch axis  $\sqrt{\frac{E}{\mu \rho_0^2}}$ . The second case is characterized by the fact that  $p_1$  and  $p_2$  are smaller than  $\sqrt{\frac{E}{\mu \rho_0^2}}$ .

We return to the first case. Setting

$$\lambda^3 (1 - \gamma p_m^2) = -k_m^4, \quad (12)$$

we rewrite Eqs. (6) and (8) in the form

$$f_m^{IV} - k_m^4 f_m = 0. \quad (13)$$

Here and henceforth  $m = 1, 2$ .

The total solutions of Eqs. (13) are written in the form

$$f_m(\theta) = C_{1m} \cos k_m \theta + C_{2m} \sin k_m \theta + C_{3m} \operatorname{ch} k_m \theta + C_{4m} \operatorname{sh} k_m \theta. \quad (14)$$

Satisfying boundary conditions (7) and (9), we get

$$\begin{aligned} f_1(\theta) &= \frac{A_1}{4\rho_0} \left( \frac{\cos k_1 \theta}{\cos k_1 \alpha} - \frac{\sin k_1 \theta}{\sin k_1 \alpha} + \frac{\operatorname{ch} k_1 \theta}{\operatorname{ch} k_1 \alpha} - \frac{\operatorname{sh} k_1 \theta}{\operatorname{sh} k_1 \alpha} \right); \\ f_2(\theta) &= \frac{A_2}{4\rho_0} \left( \frac{\cos k_2 \theta}{\cos k_2 \alpha} + \frac{\sin k_2 \theta}{\sin k_2 \alpha} + \frac{\operatorname{ch} k_2 \theta}{\operatorname{ch} k_2 \alpha} + \frac{\operatorname{sh} k_2 \theta}{\operatorname{sh} k_2 \alpha} \right). \end{aligned}$$

Thus, on conforming to Inequalities (10),  $p_1$  and  $p_2 > \sqrt{\frac{E}{\mu \rho_0^2}}$ , we get the following expression for the dimensionless function of translations  $\xi(\theta, t)$ :

$$\begin{aligned} \xi(\theta, t) &= \frac{A_1}{4\rho_0} \left( \frac{\cos k_1 \theta}{\cos k_1 \alpha} - \frac{\sin k_1 \theta}{\sin k_1 \alpha} + \frac{\operatorname{ch} k_1 \theta}{\operatorname{ch} k_1 \alpha} - \frac{\operatorname{sh} k_1 \theta}{\operatorname{sh} k_1 \alpha} \right) \sin p_1 t + \\ &+ \frac{A_2}{4\rho_0} \left( \frac{\cos k_2 \theta}{\cos k_2 \alpha} + \frac{\sin k_2 \theta}{\sin k_2 \alpha} + \frac{\operatorname{ch} k_2 \theta}{\operatorname{ch} k_2 \alpha} + \frac{\operatorname{sh} k_2 \theta}{\operatorname{sh} k_2 \alpha} \right) \sin p_2 t. \end{aligned} \quad (15)$$

We now consider the case for which Inequalities (11),  $p_1$  and  $p_2 < \sqrt{\frac{E}{\mu \rho_0^2}}$  are satisfied. Introducing the notation

$$n_m^4 = \lambda^3 (1 - \gamma p_m^2) \quad (m = 1, 2), \quad (16)$$

we rewrite Eqs. (6) and (8) in the form

$$f_m^{IV} + n_m^4 f_m = 0. \quad (17)$$

We seek the general solutions of Eqs. (17) in the form

$$\begin{aligned} f_m(\theta) &= D_{1m} \cos n_m \theta \operatorname{ch} n_m \theta + D_{2m} \sin n_m \theta \operatorname{sh} n_m \theta + \\ &+ D_{3m} \cos n_m \theta \operatorname{sh} n_m \theta + D_{4m} \sin n_m \theta \operatorname{ch} n_m \theta. \end{aligned}$$

Setting in succession  $m=1, m=2$  and satisfying accordingly boundary conditions (7) and (9), we get

$$f_1(\theta) = \frac{A_1}{2\rho_0} \left[ \frac{P_1 \cos n_1 \theta \operatorname{ch} n_1 \theta + Q_1 \sin n_1 \theta \operatorname{ch} n_1 \theta}{P_1^2 + Q_1^2} - \frac{R_1 \cos n_1 \theta \operatorname{sh} n_1 \theta + S_1 \sin n_1 \theta \operatorname{ch} n_1 \theta}{R_1^2 + S_1^2} \right];$$

$$f_2(\theta) = \frac{A_2}{2\rho_0} \left[ \frac{P_2 \cos n_2 \theta \operatorname{ch} n_2 \theta + Q_2 \sin n_2 \theta \operatorname{sh} n_2 \theta}{P_2^2 + Q_2^2} + \frac{R_2 \cos n_2 \theta \operatorname{sh} n_2 \theta + S_2 \sin n_2 \theta \operatorname{ch} n_2 \theta}{R_2^2 + S_2^2} \right],$$

where

$$\left. \begin{aligned} P_m &= \cos n_m a \operatorname{ch} n_m a; & Q_m &= \sin n_m a \operatorname{sh} n_m a; \\ R_m &= \cos n_m a \operatorname{sh} n_m a; & S_m &= \sin n_m a \operatorname{ch} n_m a; \end{aligned} \right\} \quad (18) \quad (m = 1, 2).$$

Forced vibrations of an arch in the presence of Conditions (11) will thus be governed by the following relationship:

$$\xi(\theta, t) = \frac{A_1}{2\rho_0} \left( \frac{P_1 \cos n_1 \theta \operatorname{ch} n_1 \theta + Q_1 \sin n_1 \theta \operatorname{sh} n_1 \theta}{P_1^2 + Q_1^2} - \frac{R_1 \cos n_1 \theta \operatorname{sh} n_1 \theta + S_1 \sin n_1 \theta \operatorname{ch} n_1 \theta}{R_1^2 + S_1^2} \right) \sin p_1 t +$$

$$+ \frac{A_2}{2\rho_0} \left( \frac{P_2 \cos n_2 \theta \operatorname{ch} n_2 \theta + Q_2 \sin n_2 \theta \operatorname{sh} n_2 \theta}{P_2^2 + Q_2^2} + \frac{R_2 \cos n_2 \theta \operatorname{sh} n_2 \theta + S_2 \sin n_2 \theta \operatorname{ch} n_2 \theta}{R_2^2 + S_2^2} \right) \sin p_2 t, \quad (19)$$

where  $P_m, Q_m, R_m$  and  $S_m$  are given by Formulas (18).

It is easy to show that Solutions (15) and (19) obtained above satisfy all end support conditions of the bar.

We now consider in more detail the following particular case. We assume that both supports perform synchronous vibrations. We set

$$p_1 = p_2 = p, \quad A_1 = A_2 = A;$$

which involves, according to Formulas (12) and (16), the equalities

$$k_1 = k_2 = k = \sqrt[4]{\frac{\rho_0^2}{i^2} \left( \frac{\rho_0^2 \mu}{E} p^2 - 1 \right)};$$

$$n_1 = n_2 = n = \sqrt[4]{\frac{\rho_0^2}{i^2} \left( 1 - \frac{\rho_0^2 \mu}{E} p^2 \right)}.$$

for values of  $p$  defined by the inequality  $p > \sqrt{\frac{E}{\mu \rho_0^2}}$ , we get from Eq. (15)

$$\xi(\theta, t) = \frac{A}{2\rho_0} \left( \frac{\cos k\theta}{\cos ka} + \frac{\operatorname{ch} k\theta}{\operatorname{ch} ka} \right) \sin pt. \quad (20)$$

On the other hand, for values of  $p$  obeying the inequality  $p < \sqrt{\frac{E}{\mu\rho_0^2}}$ , we get

$$\xi(\theta, t) = \frac{A}{\rho_0} \cdot \frac{P \cos n\theta \operatorname{ch} n\theta + Q \sin n\theta \operatorname{sh} n\theta}{P^2 + Q^2} \sin pt, \quad (21)$$

where

$$P = \cos na \operatorname{ch} na; \quad Q = \sin na \operatorname{sh} na.$$

We write Expression (20) in the form

$$\begin{aligned} \xi(\theta, t) = \frac{A}{2\rho_0} & \left[ \frac{\cos \theta \sqrt{\frac{\rho_0^2}{i^2} \left( \frac{\rho_0^2 \mu}{E} p^2 - 1 \right)}}{\cos \alpha \sqrt{\frac{\rho_0^2}{i^2} \left( \frac{\rho_0^2 \mu}{E} p^2 - 1 \right)}} + \right. \\ & \left. + \frac{\operatorname{ch} \theta \sqrt{\frac{\rho_0^2}{i^2} \left( \frac{\rho_0^2 \mu}{E} p^2 - 1 \right)}}{\operatorname{ch} \alpha \sqrt{\frac{\rho_0^2}{i^2} \left( \frac{\rho_0^2 \mu}{E} p^2 - 1 \right)}} \right] \sin pt. \end{aligned} \quad (22)$$

For a very large  $p$  we get

$$\xi(\theta, t) = \frac{A}{2\rho_0} \left( \frac{\cos \theta \rho_0 \sqrt{\frac{\mu p^2}{i^2 E}}}{\cos \alpha \rho_0 \sqrt{\frac{\mu p^2}{i^2 E}}} + \frac{\operatorname{ch} \theta \rho_0 \sqrt{\frac{\mu p^2}{i^2 E}}}{\operatorname{ch} \alpha \rho_0 \sqrt{\frac{\mu p^2}{i^2 E}}} \right) \sin pt$$

or

$$u(\theta, t) = \frac{A}{2} \left( \frac{\cos l \sqrt{\frac{\mu p^2}{i^2 E}}}{\cos L \sqrt{\frac{\mu p^2}{i^2 E}}} + \frac{\operatorname{ch} l \sqrt{\frac{\mu p^2}{i^2 E}}}{\operatorname{ch} L \sqrt{\frac{\mu p^2}{i^2 E}}} \right) \sin pt, \quad (23)$$

where  $l$  is the length of the arc subtended by the central angle  $\theta$ , and  $L$  is the length of the half-arch axis.

The solution of Eq. (23) is identical with the solution obtained by Ye.N. Kaporulin [3] for a straight bar, if we take  $l$  as the  $x$ -coordinate, measured from the middle of the bar.

Expression (23) can be obtained from (22) also by the limit transition

$$\alpha \rightarrow 0; \quad \theta \rightarrow 0, \quad \rho_0 \rightarrow \infty; \quad \theta \rho_0 \rightarrow l; \quad \alpha \rho_0 \rightarrow L.$$

Thus, for supports vibrating at a sufficiently high frequency (as compared with the tension-compression frequency of the bar's axis) a shallow circular arch behaves in the same manner as a straight bar of the same length.

It is known [9], [10] that the frequency of free radial vibrations of a shallow circular two-hinge arch can be represented by the formula

$$\omega_j = C_j \sqrt{\frac{E}{\mu \rho_0^2}} \quad (j = 1, 2, 3, \dots), \quad (24)$$

where

$$C_j = \sqrt{1 + j^4 \frac{\pi^4 R^2}{16 \rho_0^2 a^4}}. \quad (25)$$

Whence

$$\frac{\mu \rho_0^2}{E} = \frac{C_j^2}{\omega_j^2}. \quad (26)$$

It should be noted that ratio  $\frac{C_j^2}{\omega_j^2}$  is independent of subscript  $j$ .

Having reference to Eq. (26), we rewrite Expression (22) in a somewhat different form

$$\xi(\theta, t) = \frac{A}{2\rho_0} \left[ \frac{\cos \theta \sqrt{\frac{\rho_0^2}{i^2} \left( C_j^2 \frac{p^2}{\omega_j^2} - 1 \right)}}{\cos \alpha \sqrt{\frac{\rho_0^2}{i^2} \left( C_j^2 \frac{p^2}{\omega_j^2} - 1 \right)}} + \frac{\operatorname{ch} \theta \sqrt{\frac{\rho_0^2}{i^2} \left( C_j^2 \frac{p^2}{\omega_j^2} - 1 \right)}}{\operatorname{ch} \alpha \sqrt{\frac{\rho_0^2}{i^2} \left( C_j^2 \frac{p^2}{\omega_j^2} - 1 \right)}} \right] \sin pt.$$

We assume that  $p$ , the frequency of vibration of the supports, is the same as one of the frequencies  $\omega_j$  of free vibrations of the arch. Then

$$\xi(\theta, t) = \frac{A}{2\rho_0} \left( \frac{\cos \frac{j\pi}{2} \cdot \frac{\theta}{\alpha}}{\cos \frac{j\pi}{2}} + \frac{\operatorname{ch} \frac{j\pi}{2} \cdot \frac{\theta}{\alpha}}{\operatorname{ch} \frac{j\pi}{2}} \right) \sin pt.$$

It is obvious that synchronous vibrations of support structures induce symmetrical forced vibrations in the arch and hence  $j$  can take on odd values only.

It can be seen from the last formula that the amplitude of forced vibrations of all the points on the axis of the arch, with the exception of its ends, become infinity when  $p = \omega_j$  ( $j = 1, 3, 5, \dots$ ), i.e., we get ordinary resonance if no consideration is given to the effect of resistances.

On the basis of Formula (22) it is possible to determine the resonance conditions  $p = \omega_j$  also in the following way. It is seen from Formula (22) that the amplitude of forced vibrations becomes

infinite, if we satisfy the condition

$$\frac{\alpha^4 \rho_0^2}{i^2} \left( \frac{\rho_0^2 \mu}{E} p^2 - 1 \right) = -\frac{j^4 \pi^4}{16} \quad (j = 1, 3, 5, \dots), \quad (27)$$

which is equivalent to vanishing of the cosine in the denominator of the first term in parentheses of Formula (22). From Relationship (27) we get

$$p^2 = \frac{E}{\mu \rho_0^2} \left( 1 + j^4 \frac{\pi^4 i^2}{16 \alpha^4 \rho_0^2} \right) = \omega_j^2 \quad (j = 1, 3, 5, \dots).$$

We now point out certain features peculiar to the motion of the arch when  $p < \sqrt{\frac{E}{\mu \rho_0^2}}$ . First of all, it follows from Formulas (24) and (25) that  $p < \omega_1$  and, thus, resonance is impossible. The fact that the translations of all the points on the axis of the arch are in this case always bound, can be obtained only from Formula (21), in which the denominator cannot vanish. Since, as  $\rho_0$  increases to  $\infty$  the domain of values of  $p$  narrows down to zero, then for the given case limit transition to a straight bar is meaningless.

Let the ends of the arch perform vibrations with a very low frequency  $p$ . We set

$$n^4 = \lambda^4 (1 - \gamma p^2) \approx \lambda^4.$$

Then

$$\xi(\theta, t) = \frac{A}{\rho_0} \cdot \frac{P_* \cos \theta \sqrt{\lambda} \operatorname{ch} \theta \sqrt{\lambda} + Q_* \sin \theta \sqrt{\lambda} \operatorname{sh} \theta \sqrt{\lambda}}{P_*^2 + Q_*^2} \sin pt,$$

where

$$P_* = \cos \alpha \sqrt{\lambda} \operatorname{ch} \alpha \sqrt{\lambda}; \quad Q_* = \sin \alpha \sqrt{\lambda} \operatorname{sh} \alpha \sqrt{\lambda}.$$

Combining Solutions (15) and (19), it is possible to obtain other particular cases of the problem at hand.

Let us now clarify the feasibility of approximate solution of the problem at hand by the Bubnov-Galerkin method. If the approximating function  $\xi(\theta, t)$  is properly chosen, this will make it possible, first of all, to subject the translation of supports to more general laws, and secondly, will make possible, without changing the methods of solution, to enter the nonlinear domain and thus clarify those points which were lost by linearizing the basic equation (1). We assume that the hinged ends of the arch perform vertical motions described by functions  $\Phi(t)$  and  $\Psi(t)$ . Selecting as the approximating function of dimensionless deflections

$$\xi(\theta, t) = \frac{\Phi(t)}{\rho_0} + \frac{\Psi(t) - \Phi(t)}{2\rho_0 \alpha} (\alpha - \theta) + \sum E_j(t) \sin \frac{j\pi(\alpha - \theta)}{2\alpha}, \quad (28)$$

we see that

$$\xi(+\alpha, t) = \frac{\Phi(t)}{\rho_0}; \quad \xi(-\alpha, t) = \frac{\Psi(t)}{\rho_0}; \quad \left( \frac{\partial \xi}{\partial \theta} \right)_{\theta = \pm \alpha} = 0. \quad (29)$$

These conditions are the boundary conditions for the problem at hand. Functions  $E_j(t)$  are to be determined. Expressions for ap-

proximating functions, similar to (28), were used by K.S. Kolesnikov [7] in the study of vibrations of straight bars with non-steadily supported ends.

We return again to the case for which an exact solution was previously found. Thus, we assume that the ends of the arch move according to the equations

$$\Phi(t) = A \sin pt; \Psi(t) = A \sin pt. \quad (30)$$

Having reference to Eq. (30), we rewrite Expression (28) in the form

$$\xi(\theta, t) = \frac{A}{\rho_0} \sin pt + \sum E_j(t) \sin \frac{j\pi(a-\theta)}{2a}. \quad (31)$$

If we restrict ourselves to the first term of the series and apply the Bubnov-Galerkin method to the solution of Eq. (1), then we get

$$\int_{-\pi}^{+\pi} \left[ E_1 \left( \frac{\pi}{2a} \right)^4 \cos \frac{\pi\theta}{2a} - \lambda^2 \left( \frac{A}{\rho_0} \sin pt + E_1 \cos \frac{\pi\theta}{2a} \right) \left( \frac{\pi}{2a} \right)^2 E_1 \cos \frac{\pi\theta}{2a} + \right. \\ \left. + \lambda^2 \left( \frac{A}{\rho_0} \sin pt + E_1 \cos \frac{\pi\theta}{2a} \right) + \lambda^2 \gamma \left( -\frac{A\rho^3}{\rho_0} \sin pt + \ddot{E}_1 \cos \frac{\pi\theta}{2a} \right) \cos \frac{\pi\theta}{2a} \right] d\theta = 0.$$

After integration and elementary algebraic transformations, we will get the following ordinary differential equation for  $E_1(t)$

$$\ddot{E}_1 + \omega^2 \left( 1 - \frac{A\pi^2}{4a^2\rho_0\gamma\omega^2} \sin pt \right) E_1 - \frac{2\pi}{3a^2\gamma} E_1^2 = \frac{4A}{\pi\rho_0} \left( \rho^3 - \frac{1}{\gamma} \right) \sin pt, \quad (32)$$

where  $\ddot{E}_1$  denotes the second derivative with respect to time, while  $\omega$  denotes the fundamental frequency of free radial vibrations [see Formula (24)].

We note the following two particular forms of Eq. (32):

$$\ddot{E}_1 + \omega^2 E_1 = \frac{4A}{\pi\rho_0} \left( \rho^3 - \frac{1}{\gamma} \right) \sin pt; \quad (33)$$

$$\ddot{E}_1 + \omega^2 \left( 1 - \frac{A\pi^2}{4a^2\rho_0\gamma\omega^2} \sin pt \right) E_1 = \frac{4A}{\pi\rho_0} \left( \rho^3 - \frac{1}{\gamma} \right) \sin pt. \quad (34)$$

We first of all note that Eq. (33) can be obtained independently only by the Bubnov-Galerkin method from Eq. (2); here, if we do not restrict ourselves to the first term of the series contained in Expression (31), then instead of the single equation (33), we get the following ensemble of equations

$$\left. \begin{aligned} \ddot{E}_1 + \omega_1^2 E_1 &= \frac{4A}{\pi\rho_0} \left( \rho^3 - \frac{1}{\gamma} \right) \sin pt; \\ \ddot{E}_2 + \omega_2^2 E_2 &= 0; \\ \ddot{E}_3 + \omega_3^2 E_3 &= \frac{4A}{3\pi\rho_0} \left( \rho^3 - \frac{1}{\gamma} \right) \sin pt; \\ \ddot{E}_4 + \omega_4^2 E_4 &= 0. \end{aligned} \right\} \quad (35)$$

Solving these equations, and having reference only to forced vibrations, we get

$$\xi(\theta, t) = \frac{A}{\rho_0} \sin pt + \frac{4A}{\pi \rho_0} \left(1 - \frac{1}{\gamma p^2}\right) \times \\ \times \left[ \frac{\cos \frac{\pi \theta}{2\alpha}}{\frac{\omega_1^2}{p^2} - 1} - \frac{\cos \frac{3\pi \theta}{2\alpha}}{3 \left(\frac{\omega_3^2}{p^2} - 1\right)} + \frac{\cos \frac{5\pi \theta}{2\alpha}}{5 \left(\frac{\omega_5^2}{p^2} - 1\right)} - \dots \right] \sin pt.$$

We represent the last equality in the form

$$\xi(\theta, t) = V \frac{A}{\rho_0} \sin pt,$$

where  $V$  denotes the expression

$$V = 1 + \frac{4}{\pi} \left(1 - \frac{1}{\gamma p^2}\right) \left[ \frac{\cos \frac{\pi \theta}{2\alpha}}{\frac{\omega_1^2}{p^2} - 1} - \frac{\cos \frac{3\pi \theta}{2\alpha}}{3 \left(\frac{\omega_3^2}{p^2} - 1\right)} + \frac{\cos \frac{5\pi \theta}{2\alpha}}{5 \left(\frac{\omega_5^2}{p^2} - 1\right)} - \dots \right].$$

Calculations show that when using in our problem approximating function (31), a satisfactory agreement of the exact and approximation solutions is obtained for sufficiently shallow and not very flexible arches. To clarify the above, we present in abbreviated form a table of the amplitude coefficient  $V$  of forced vibrations of the middle point of the arch ( $\theta = 0$ ); the parentheses contain exact values of this coefficient obtained from Formula (20).

$\frac{l^2}{\rho_0^2}$	$2\alpha$ in °	$\frac{1}{p^2 \gamma} = 0.01$
0,01	20	-2,799 (-2,924)
	15	2,155 ( 2,141)
0,0004	15	-0,586 (-0,459)
	10	-1,25 (-1,25)
0,0001	10	-0,563 (-0,479)
	8	-0,78 (-0,75)

1) In.

It is obvious that by specifying ratios  $\frac{\rho_0^2}{l^2}$  and  $\frac{1}{p^2 \gamma}$  we automatically establish a given relationship between frequencies  $p$  and  $\omega_j$ . Thus, for example, for an arch with central angle  $2\alpha = 20^\circ$  and for  $\frac{\rho_0^2}{l^2} = 10000$  and  $\frac{1}{\gamma p^2} = 0,1$  we will get  $\omega_1^2 = 0,0167 p^2$ ;  $\omega_3^2 = 0,541 p^2$ ;  $\omega_5^2 = 4,111 p^2$ , etc. On the other hand, by keeping these ratios constant it is easy to find the values of central angle  $2\alpha$  for which resonance will set on. In the example just considered  $V$  goes to infinity if

$$2\alpha_1 \approx 5^\circ 40'; 2\alpha_2 \approx 16^\circ 40'; 2\alpha_3 \approx 28^\circ 32'.$$

The above results are reliable to one or another degree only for domains far removed from resonance, since the fact that various kinds of resistances were not taken here into account when solving the problem radically affects the equations of motion proper as well as their solutions.

We return to Eqs. (33) and (34). They have in common the fact that both are linear, the first of them has constant and the other has periodically varying coefficients. We recall that Eq. (33) can be regarded as the result of a problem initially stated in the linear form, while Eq. (34) appeared already as a result of linearization of Eq. (32). We have seen that the solution of Eqs. (33) and (35) appreciably schematize the solution of our problem. As to Eq. (34), it can be judged by its external appearance that, on the one hand, the complexity of the study has increased, and, on the other, that it makes it possible to discover certain specific properties of the bar's motion.

Thus, depending on the stage in solving the problem at which we leave the nonlinear domain and start treating the problem linearly, it is possible to retain or drop substantial characteristics of the phenomenon under study.

We shall restrict ourselves from now on to the consideration of one particular but at the same time characteristic case. We set

$$\rho = \frac{1}{\gamma} = \sqrt{\frac{E}{\mu \rho_0^2}}, \quad (36)$$

i.e., we shall assume that the frequency of support vibrations is identical with the frequency of tension-compression vibrations of the axis of the arch. We have seen earlier that this case had corresponding to it motion of the arch as a single undeformed body (if free vibrations are not considered). It is obvious that, when Condition (36) is satisfied, we will always have the inequality

$$\rho < \omega_j.$$

Equations (33) and (34) will in this case become the following equations:

$$\begin{aligned} \ddot{E}_1 + \omega^2 F &= 0; \\ \ddot{E}_1 + \omega^2 \left( 1 - \frac{A\pi^2}{4\alpha^2 \rho_0 \gamma \omega^2} \sin \rho t \right) E_1 &= 0. \end{aligned} \quad (37)$$

The first of these equations has become an equation of the free vibrations of a harmonic oscillator and, consequently, Relationship (36) can be regarded as a condition eliminating the possibility of appearance of forced vibrations of an arch attendant to the vibration of supports according to (30). We arrive at the same conclusion from the solution of the problem which is formulated linearly from the very beginning. We obtain another solution if we turn to Eq. (37). This equation can be converted by ordinary methods into a Mathieu equation and thus it would become possible to study features in the vibrations of the arch which are related

to the stable and unstable regime of motion.

In order to transform Eq. (37) to the standard form of the Mathieu equation, we replace the independent variable  $t$  by a new dimensionless  $\tau$ , using the substitution

$$\rho t + \frac{\pi}{2} = 2\tau,$$

then  $\ddot{E}_1 = \frac{\rho^2}{4} E_1'$ ; the primes denote derivatives with respect to  $\tau$ . After elementary transformations we get

$$E'' + (a - 2q \cos 2\tau) E_1 = 0,$$

where

$$a = 4 \frac{\omega^2}{\rho^2}; \quad 2q = \frac{A \pi^2}{\rho_0 a^2}.$$

It is known from the general theory of the Mathieu equation that "...the solutions are basically stable when  $a > 0$  and  $|2q| < a$  and are basically unstable outside of this range, as was predetermined..." [11, page 368]. On the basis of this general conclusion we can set up for our problem a sufficiently simple condition for stable vibrations. It follows from the inequality  $2q < a$  that

$$\frac{A}{\rho_0} < 4 \frac{\omega^2}{\rho^2} \left( \frac{a}{\pi} \right)^2$$

or, in another version, by having reference to Eq. (24)

$$\frac{A}{\rho_0} < 4 \left( \frac{a}{\pi} \right)^2 \left( 1 + \frac{l^2 \pi^4}{16 \rho_0^2 a^4} \right). \quad (38)$$

On the basis of solely practical considerations, parameters  $l$ ,  $\rho_0$  and  $a$  are usually assigned values for which the stability criterion (38) is reliably satisfied. Thus, for example, for  $\frac{l^2}{\rho_0^2} = 0.0001$  and  $2\alpha = 10^\circ$ , we get  $\frac{A}{\rho_0} < 0.315$ .

Thus, if force factors are not considered as being the direct causes of vibrations, it is possible to take specified translations of individual members of a system as sources of vibration processes. Being maintained for a given time interval, these disturbances may, under known conditions, produce in the entire system vibrations with resonance of the ordinary (force) and parametric types. This kinematic approach to the study of vibrations processes, which was used in the present paper, can be found to be particularly suitable in cases when it is possible to observe the motion of individual characteristic points (cross sections) of the bar, plate or other vibrating body. Ordinary or force resonance is found in this approach to the study of vibrations without additional analysis of forces affecting the system.

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## DESIGN OF FRAMES WITH PONDERABLE ELEMENTS, CARRYING TWO CONCENTRATED MASSES EACH, FOR A VIBRATING LOAD, USING THE METHOD OF TRANSLATIONS

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The unknowns in designing frames by the method of translations are the angular and linear translations of the system's joints. Standard elements of the main system are statically indeterminate bars with different end supports. The present paper derives formulas of dynamic reactions of joints of different bars of a main system with a uniformly distributed mass and with two concentrated masses, due to vibration loads and various unit vibration translations (see below). These formulas are needed for setting up canonical equations of the method of translations and for further design of frames for vibrating loads for the nondeformed state (for the nondeformed scheme).

The dynamic reactions of vibration effects with frequency  $\theta$  were determined from equations of the method of initial parameters

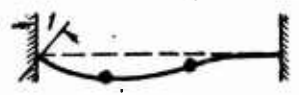
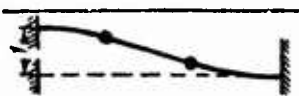
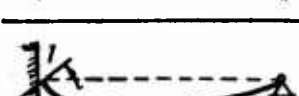

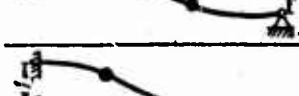
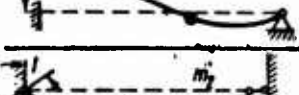
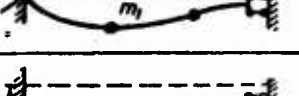
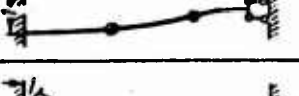
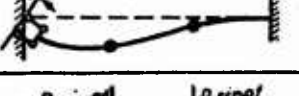
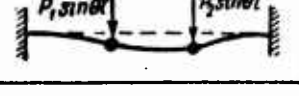
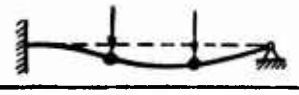
$$\begin{aligned} y(x) &= y_0 A_{kx} + \frac{y'_0}{k} B_{kx} - \frac{M_0}{k^2 EJ} C_{kx} - \frac{Q_0}{k^3 EJ} D_{kx} - \frac{1}{EI} (A_{kx} - 1); \\ y'(x) &= ky_0 D_{kx} + y'_0 A_{kx} - \frac{M_0}{k EJ} \theta_{kx} - \frac{Q_0}{k^2 EJ} C_{kx} - \frac{q_0}{k^3 EJ} D_{kx}; \\ M(x) &= -k^2 EJ y_0 C_{kx} - k EJ y'_0 D_{kx} + M_0 A_{kx} + \frac{Q_0}{k} B_{kx} + \frac{q_0}{k^2} C_{kx}; \\ Q(x) &= -k^3 EJ y_0 B_{kx} - k^2 EJ y'_0 C_{kx} + k M_0 D_{kx} + Q_0 A_{kx} + \frac{q_0}{k} B_{kx}, \end{aligned}$$

where  $y_0, y'_0, M_0, Q_0, q_0$  are, respectively, the deflection, angle of rotation, bending moment, shearing force and the distributed load at the left end of the bar;  $A_{kx}, B_{kx}, C_{kx}$  and  $D_{kx}$  are A.N. Krylov's functions;  $k^4 = \frac{m\theta^2}{EJ}$ ;  $m$  is the mass per unit length of bar.

The inertia forces produced by concentrated masses were taken into account in the form of concentrated forces  $R = m\theta^2 y^*$ , where  $m$  is the concentrated mass and  $y^*$  is its coordinate.

We briefly illustrate the process of solution using an example of a unit vibratory rotation of the left end of a bar clamped at both ends (see the first case in the table below).

TABLE

	$M_0$	$Q_0$	$M_L$	$Q_L$
	$\Phi_1 = \frac{U_1}{V_1}$	$\Phi_2 = \frac{U_2}{V_1}$	$\Phi_3 = \frac{U_3}{V_1}$	$\Phi_4 = \frac{U_4}{V_1}$
	$\Phi_5 = \frac{U_5}{V_1}$	$\Phi_6 = \frac{U_6}{V_1}$	$\Phi_7 = \frac{U_7}{V_1}$	$\Phi_8 = \frac{U_8}{V_1}$
	$\Phi_9 = \frac{U_9}{V_2}$	$\Phi_{10} = \frac{U_{10}}{V_2}$	0	$\Phi_{15} = \frac{U_{15}}{V_3}$
	$\Phi_{11} = \frac{U_{11}}{V_2}$	$\Phi_{12} = \frac{U_{12}}{V_2}$	0	$\Phi_{16} = \frac{U_{16}}{V_3}$
	$\Phi_{13} = \frac{U_{13}}{V_2}$	$\Phi_{14} = \frac{U_{14}}{V_2}$	0	$\Phi_{17} = \frac{U_{17}}{V_3}$
	$\Phi_{18} = \frac{U_{18}}{V_4}$	$\Phi_{19} = \frac{U_{19}}{V_4}$	$\Phi_{23} = \frac{U_{23}}{V_5}$	0
	$\Phi_{20} = \frac{U_{20}}{V_4}$	$\Phi_{21} = \frac{U_{21}}{V_4}$	$\Phi_{26} = \frac{U_{26}}{V_5}$	0
	$\Phi_{22} = \frac{U_{22}}{V_5}$	0	$\Phi_{24} = \frac{U_{24}}{V_4}$	$\Phi_{25} = \frac{U_{25}}{V_4}$
	$\Phi_{27} = \frac{U_{27}}{V_1}$	$\Phi_{28} = \frac{U_{28}}{V_1}$	$-\Phi_{27}$	$-\Phi_{28}$
	$\Phi_{29} = \frac{U_{29}}{V_2}$	$\Phi_{30} = \frac{U_{30}}{V_2}$	0	$\Phi_{31} = \frac{U_{31}}{V_3}$
	$\Phi_{32} = \frac{U_{32}}{V_4}$	$\Phi_{33} = \frac{U_{33}}{V_4}$	$\Phi_{34} = \frac{U_{34}}{V_5}$	0

Notes: 1. In defining  $M_L$  and  $Q_L$  in formulas for determining  $u$  and  $v$ ,  $m_1$  should be replaced by  $m_2$ , and subscripts  $a$  should be replaced by  $l - b$ , and  $b$  should be replaced by  $l - a$ .

2. The formulas for  $M_0$  and  $Q_0$  presented in the table can also be used if the supports with the corresponding translations change places (mass  $m_1$  then being to the left).

On taking into account that the boundary conditions at the left end of the bar  $y_0=0, y'_0=1$ , the equations of the method of initial parameters will be written as follows for the individual sections:

1st section:  $0 \leq x < a$

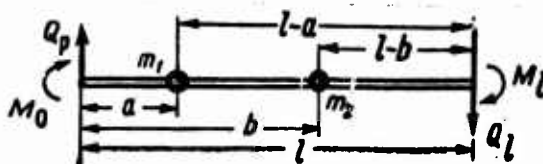
$$y_1(x) = \frac{B_{kx}}{k} - \frac{M_0}{k^3 EJ} C_{kx} - \frac{Q_0}{k^2 EJ} D_{kx}; \quad y_1'(x) = A_{kx} - \frac{M_0}{k EJ} B_{kx} - \frac{Q_0}{k^2 EJ} C_{kx};$$

2nd section:  $a \leq x < b$

$$y_2(x) = y_1(x) + \frac{m_1 \theta^3 y(a)}{k^3 EJ} D_{k(x-a)}; \quad y_2'(x) = y_1'(x) + \frac{m_1 \theta^3 y(a)}{k^3 EJ} C_{k(x-a)};$$

3rd section:  $b \leq x < l$

$$y_3(x) = y_2(x) + \frac{m_2 \theta^3 y(b)}{k^3 EJ} D_{k(x-b)}; \quad y_3'(x) = y_2'(x) + \frac{m_2 \theta^3 y(b)}{k^3 EJ} C_{k(x-b)}.$$



Satisfying the boundary conditions at the right end: when  $x = l$ ,  $y_l = 0$  and  $y_l' = 0$ , we get the equations for the unknown initial parameters  $M_0$  and  $Q_0$ . The values of  $M_0$  and  $Q_0$  for this and other cases which were considered are presented in the table.

In the study of free vibrations of frames the load terms of the canonical equations should be set equal to zero, while frequency  $\theta$  should be considered as the unknown frequency of free vibrations.

The notation used is:

$$a_l = A_{kl}; \quad a_{l,a} = A_{k(l-a)}; \quad a_{b,a} = A_{k(b-a)}; \quad b_l = B_{kl}; \quad b_{l,a} = B_{k(l-a)} \text{ etc.}$$

$$M_1 = \frac{m_1 \theta^3}{k^3 EJ}; \quad M_2 = \frac{m_2 \theta^3}{k^3 EJ}; \quad M = \frac{m_1 m_2 \theta^4}{k^6 E^2 J^2};$$

$$A(a) = c_l d_{l,a} - c_{l,a} d_l; \quad A(b) = c_l d_{l,b} - c_{l,b} d_l;$$

$$B(a) = c_{l,a} c_l - b_l d_{l,a}; \quad B(b) = c_{l,b} c_l - b_l d_{l,b};$$

$$C = c_{l,a} d_{l,b} - c_{l,b} d_{l,a};$$

$$D(a) = b_l c_{l,a} - a_l d_{l,a}; \quad E(a) = d_{l,a} d_l - a_l c_{l,a};$$

$$F(a) = b_l d_{l,a} - b_{l,a} d_l; \quad K(a) = b_{l,a} c_l - a_l d_{l,a};$$

$$L = b_{l,a} d_{l,b} - b_{l,b} d_{l,a};$$

$$\begin{aligned}
M(a) &= b_{l,a} b_l - d_{l,a} d_l; \quad N(a) = c_l d_{l,a} - a_l b_{l,a}; \\
P(a) &= a_l c_{l,a} - c_{l,a} c_l; \quad Q(a) = a_{l,a} b_l - c_{l,a} d_l; \\
R &= a_{l,a} c_{l,b} - a_{l,b} c_{l,a}; \quad S(a) = a_{l,a} a_l - c_{l,a} c_l. \\
U_1 &= \frac{1}{k^2} \{ b_l c_l - a_l d_l + M_1 [b_a A(a) + d_a D(a)] + M_2 [b_b A(b) + d_b D(b)] + \\
&\quad + M [b_a d_{b,a} A(b) + d_a d_{b,a} D(b) + C(b_b d_a - b_a d_b)] \}; \\
U_2 &= \frac{1}{k} \{ a_l c_l - b_l^2 + M_1 [b_a B(a) - c_a D(a)] + M_2 [b_b B(b) - c_b D(b)] + \\
&\quad + M [b_a d_{b,a} B(b) - c_a d_{b,a} D(b) + C(b_a c_b - b_b c_a)] \}; \\
U_3 &= \frac{1}{k^2} (d_l + M_1 d_a d_{l,a} + M_2 d_b d_{l,b} + M d_a d_{l,b} d_{b,a}); \\
U_4 &= \frac{1}{k} (c_l + M_1 c_a d_{l,a} + M_2 c_b d_{l,b} + M c_a d_{l,b} d_{b,a}); \\
U_5 &= \frac{1}{k} \{ d_l^2 - a_l c_l + M_1 [d_a E(a) - a_a A(a)] + M_2 [d_b E(b) - a_b A(b)] + \\
&\quad + M [d_a d_{b,a} E(b) - a_a d_{b,a} A(b) + C(a_a d_b - a_b d_a)] \}; \\
U_6 &= a_l b_l - c_l d_l - M_1 [a_a B(a) + c_a E(a)] - M_2 [a_b B(b) + c_b E(b)] - \\
&\quad - M [a_a d_{b,a} B(b) + c_a d_{b,a} E(b) + C(a_a c_b - a_b c_a)]; \\
U_7 &= -\frac{1}{k} (c_l + M_1 c_{l,a} d_a + M_2 c_{l,b} d_b + M c_{l,b} d_a d_{b,a}); \\
U_8 &= b_l + M_1 c_a c_{l,a} + M_2 c_b c_{l,b} + M c_a c_{l,b} d_{b,a}; \\
U_9 &= \frac{1}{k^2} \{ b_l^2 - d_l^2 + M_1 [b_a F(a) + d_a M(a)] + M_2 [b_b F(b) + d_b M(b)] + \\
&\quad + M [b_a d_{b,a} F(b) + d_a d_{b,a} M(b) + L(b_b d_a - b_a d_b)] \}; \\
U_{10} &= \frac{1}{k} \{ c_l d_l - a_l b_l + M_1 [b_a K(a) - c_a M(a)] + M_2 [b_b K(b) - c_b M(b)] + \\
&\quad + M [b_a d_{b,a} K(b) - c_a d_{b,a} M(b) + L(b_a c_b - b_b c_a)] \}; \\
U_{11} &= -\frac{1}{k} (b_l + M_1 b_{l,a} d_a + M_2 b_{l,b} d_b + M b_{l,b} d_a d_{b,a}); \\
U_{12} &= a_l + M_1 b_{l,a} c_a + M_2 b_{l,b} c_b + M b_{l,b} c_a d_{b,a}; \\
U_{13} &= \frac{1}{k} \{ c_l d_l - a_l b_l + M_1 [d_a N(a) - a_a F(a)] + M_2 [d_b N(b) - a_b F(b)] + \\
&\quad + M [d_a d_{b,a} N(b) - a_a d_{b,a} F(b) + L(a_a d_b - a_b d_a)] \}; \\
U_{14} &= a_l^2 - c_l^2 - M_1 [a_a K(a) + c_a N(a)] - M_2 [a_b K(b) + c_b N(b)] - \\
&\quad - M [a_a d_{b,a} K(b) + c_a d_{b,a} N(b) + L(a_a c_b - a_b c_a)]; \\
U_{15} &= \frac{1}{k} (b_l + M_1 b_a d_{l,a} + M_2 b_b d_{l,b} + M b_a d_{l,b} d_{b,a}); \\
U_{16} &= a_l^2 - b_l d_l - M_1 [a_a D(a) + b_a E(a)] - M_2 [a_b D(b) + b_b E(b)] - \\
&\quad - M [a_a d_{b,a} D(b) + b_a d_{b,a} E(b) + C(a_a b_b - a_b b_a)]; \\
U_{17} &= -a_l - M_1 b_a c_{l,a} - M_2 b_b c_{l,b} - M b_a c_{l,b} d_{b,a};
\end{aligned}$$

$$\begin{aligned}
U_{18} &= a_i^2 - c_i^2 + M_1 [b_a P(a) + d_a S(a)] + M_2 [b_b P(b) + d_b S(b)] + \\
&\quad + M [b_a d_{b,a} P(b) + d_a d_{b,a} S(b) + R(b_b d_a - b_a d_b)]; \\
U_{19} &= k \{b_i c_i - a_i d_i + M_1 [b_a Q(a) - c_a S(a)] + M_2 [b_b Q(b) - c_b S(b)] + \\
&\quad + M [b_a d_{b,a} Q(b) - c_a d_{b,a} S(b) + R(b_a c_b - b_b c_a)]\}; \\
U_{20} &= k \{a_i d_i - b_i c_i + M_1 [a_a P(a) + d_a K(a)] + M_2 [a_b P(b) + d_b K(b)] + \\
&\quad + M [a_a d_{b,a} P(b) + d_a d_{b,a} K(b) + R(a_b d_a - a_a d_b)]\}; \\
U_{21} &= k^2 \{b_i^2 - d_i^2 + M_1 [a_a N(a) - c_a K(a)] + M_2 [a_b N(b) - c_b K(b)] + \\
&\quad + M [a_a d_{b,a} N(b) - c_a d_{b,a} K(b) + R(a_a c_b - a_b c_a)]\}; \\
U_{22} &= a_i^2 - o_i d_i - M_1 [a_a D(a) + b_a E(a)] - M_2 [a_b D(b) + b_b E(b)] - \\
&\quad - M [a_a d_{b,a} D(b) + b_a d_{b,a} E(b) + C(a_a b_a - a_b b_a)]; \\
U_{23} &= a_i + M_1 a_a d_{i,a} + M_2 a_b d_{i,b} + M a_a d_{i,b} d_{b,a}; \\
U_{24} &= -a_i - M_1 a_{i,a} d_a - M_2 a_{i,b} d_b - M a_{i,b} d_a d_{b,a}; \\
U_{25} &= k(d_i + M_1 a_{i,a} c_a + M_2 a_{i,b} c_b + M a_{i,b} c_a d_{b,a}); \\
U_{26} &= k(d_i + M_1 a_a c_{i,a} + M_2 a_b c_{i,b} + M a_a c_{i,b} d_{b,a}); \\
U_{27} &= \frac{\sin \theta t}{k^4 EJ} [P_1 A(a) + P_2 A(b) + C(M_1 P_2 d_a - M_2 P_1 d_b) + M_2 P_1 d_{b,a} A(b)]; \\
U_{28} &= \frac{\sin \theta t}{k^3 EJ} [P_1 B(a) + P_2 B(b) + C(M_2 P_1 c_b - M_1 P_2 c_a) + M_2 P_1 d_{b,a} B(b)]; \\
U_{29} &= \frac{\sin \theta t}{k^4 EJ} [P_1 F(a) + P_2 F(b) + L(M_1 P_2 d_a - M_2 P_1 d_b) + M_2 P_1 d_{b,a} F(b)]; \\
U_{30} &= \frac{\sin \theta t}{k^3 EJ} [P_1 K(a) + P_2 K(b) + L(M_2 P_1 c_b - M_1 P_2 c_a) + M_2 P_1 d_{b,a} K(b)]; \\
U_{31} &= -\frac{\sin \theta t}{k^3 EJ} [P_1 D(a) + P_2 D(b) + C(M_2 P_1 b_b - M_1 P_2 b_a) + M_2 P_1 d_{b,a} D(b)]; \\
U_{32} &= \frac{\sin \theta t}{k^3 EJ} [P_1 P(a) + P_2 P(b) + R(M_1 P_2 d_a - M_2 P_1 d_b) + M_2 P_1 d_{b,a} P(b)]; \\
U_{33} &= \frac{\sin \theta t}{k EJ} [P_1 Q(a) + P_2 Q(b) + R(M_2 P_1 c_b - M_1 P_2 c_a) + M_2 P_1 d_{b,a} Q(b)]; \\
U_{34} &= -\frac{\sin \theta t}{k^3 EJ} [P_1 E(a) + P_2 E(b) + C(M_1 P_2 a_a - M_2 P_1 a_b) + M_2 P_1 d_{b,a} E(b)]; \\
V_1 &= \frac{1}{k^3 EJ} \{c_i^2 - b_i d_i + M_1 [c_a A(a) + d_a B(a)] + M_2 [c_b A(b) + d_b B(b)] + \\
&\quad + M [c_a d_{b,a} A(b) + d_a d_{b,a} B(b) + C(c_b d_a - c_a d_b)]\};
\end{aligned}$$

$$V_2 = \frac{1}{k^2 EJ} \{ b_1 c_1 - a_1 d_1 + M_1 [c_a F(a) + d_a K(a)] + M_2 [c_b F(b) + d_b K(b)] + \\ + M [c_a d_{b,a} F(b) + d_a d_{b,a} K(b) + L (c_b d_a + c_a d_b)] \};$$

$$V_3 = \frac{1}{k^2 EJ} \{ a_1 d_1 - b_1 c_1 - M_1 [b_a A(a) + d_a D(a)] - M_2 [b_b A(b) + d_b D(b)] - \\ - M [b_a d_{b,a} A(b) + d_a d_{b,a} D(b) - C (b_b d_a - b_a d_b)] \};$$

$$V_4 = \frac{1}{k EJ} \{ a_1 b_1 - c_1 d_1 + M_1 [c_a P(a) + d_a Q(a)] + M_2 [c_b P(b) + d_b Q(b)] + \\ + M [c_a d_{b,a} P(b) + d_a d_{b,a} Q(b) + R (c_b d_a - c_a d_b)] \};$$

$$V_5 = \frac{1}{k EJ} \{ a_1 b_1 - c_1 d_1 - M_1 [a_a B(a) + c_a E(a)] - M_2 [a_b B(b) + c_b E(b)] - \\ - M [a_a d_{b,a} B(b) + c_a d_{b,a} E(b) + C (a_a c_b - a_b c_a)] \}.$$

## ON DESIGNING MINIMUM-WEIGHT BAR SYSTEMS

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### §1. Method of Designing Minimum-Weight Statically Indeterminate Trusses on the Basis of the Limiting State

The problem of designing a minimum-weight truss is treated here not from the most general point of view, which includes in the concept of design seeking in the first place the *system* of the truss, but from the narrower point of view of selecting the cross sections of the bars on the condition that all the remaining dimensions are specified. The truss is regarded hinged, the material's deformation is considered as obeying the Prandtl diagram, the load is assumed to be constant, applied only in the joints of the truss, no consideration is given to longitudinal flexure. The term limiting state is defined as such for which the truss becomes a mechanism.

The problem consists of two parts:

1) selecting cross sections, ensuring the previously specified, upon the designer's consideration, stress distribution between the bars of the truss;

2) finding among all the trusses with the specified stress distribution that which would have minimum weight.

1. If the scheme of the truss and the cross-sectional areas of its bars are specified, then determination of stresses produced by the given load can be called a direct problem. Determination of the coefficient  $k$  by which it is necessary to multiply the given load to convert it into the limiting load for the truss under consideration, can also be regarded as a direct problem. Both problems are obviously determinate, allowing for a unique solution. It is possible to specify exactly the entire sequence of the appearance of yielding in the bars on gradual increase in the load up to the load for which the truss becomes a mechanism.

The inverse problem consists in selecting the cross sections of all the bars for a given limiting load. Reference is not had here to a passive checking of the existing force properties of the truss, but to "commanding" these properties. This problem, unlike the direct problem, may have an infinite multitude of solutions. Here will be pointed out some characteristic features of this problem as compared with the direct and inverse problem pertain-

ing to design in the *elastic* stage.

In designing trusses as elastic systems using the method of specified stresses, it is sufficient to specify forces  $X_i$  in the redundant bars and the stresses in the principal bars; all the cross-sectional areas are thus determined. In designing on the basis of the limiting state a new problem arises, i.e., it becomes necessary to select the failure mechanism which sets on before all the other possible mechanisms attendant to gradual increases in the load and corresponding inclusion of bars in the state of yielding. If the truss contains  $n$  redundant bars, then for converting it into a mechanism the yielding should embrace either  $n + 1$  bars, or in individual cases less than this number. In the general case some  $n$  redundant and 1 principal bar should be yielding.

The designer can select a failure mechanism from among a large number of those theoretically possible. Let the number of bars in the truss be  $m$ . If it is statically determinate, then each bar is absolutely necessary; consequently, the number of possible mechanisms is  $m$ . It is sufficient to assign to any of the bars a cross-sectional area  $F_i < \frac{N_i}{\sigma_0}$ , where  $\sigma_0$  is the yield strength, and to assign to all the other bars areas for which the stresses do not reach the value  $\sigma_0$ , and the goal is thus achieved.

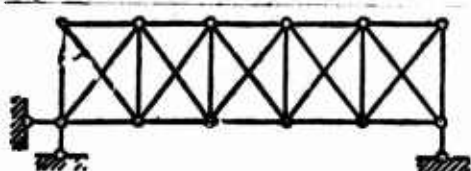


Fig. 1

If the truss is statically indeterminate, but among its bars it has  $m_1$  which are *absolutely necessary*, then  $m_1$  mechanisms which are formed by yielding of any of these bars are the elementary failure mechanisms. In addition to these mechanisms are possible others, produced by yielding of the statically indeterminate (conventionally necessary) bars.

Let the bar contain  $m$  conventionally necessary bars, including  $n$  redundant bars. The number of mechanisms due to yielding of the conventionally necessary bars in any case cannot be less than the number of combinations of  $m$  elements taken  $n + 1$  at a time, i.e., the number

$$\frac{m(m-1) \dots (m-n)}{(n+1)!}$$

This is a tremendous number. For example, for a 5-panel truss (Fig. 1), for which  $m=26$ ,  $n=5$ , the number of possible mechanisms exceeds 230,230. Even for a truss with one redundant bar the number of mechanisms cannot be less than  $\frac{m(m-1)}{2}$ ; for a statically determinate truss it is  $m$ .

2. One of the central points in designing statically indeterminate trusses on the basis of the limiting state is selection of those bars whose yielding will convert the truss into a mechanism.

For this reason it is inexpedient here to completely follow the ways which have justified themselves for calculations in the elastic range. In particular, it is inexpedient to start with specifying forces  $X_i$  in the redundant bars. One of the possible computational procedures is as follows.

a) Select the main, statically determinate system from the total number of  $\frac{m(m-1) \dots (m-n+1)}{n!}$  possible; load it by the given limiting load (which also includes the approximate proper weight of the truss) and determine the signs of forces  $N_p$  of all of its bars. The determination of force  $N_p$  proper is not mandatory at this stage.

b) Seek the solution of the problem on the condition that the signs of forces in these bars will be the same also in the actual truss at the time it reaches its limiting state. For this

c) without determining the final forces, specify the corresponding stresses in the bars, i.e.,  $n + 1$  stresses equal to  $\sigma_0$  and remaining stresses of somewhat lower magnitude.

Calculate the corresponding elongations or foreshortening of the bars from the formula  $\Delta_i = \frac{\sigma_i}{E} l_i$ . The signs of the stresses should be taken according to those of "c".

Instead of specifying stresses  $\sigma_i$  and then calculating the changes in length  $\Delta_i$ , it is possible to specify the latter immediately, i.e., in  $n + 1$  bars  $\Delta_i$  should have the limiting value for elastic deformations  $\frac{\sigma_0}{E} l_i$ , while in the balance of the bars their value should be somewhat less than the limiting. Here the designer remains free in selecting  $n + 1$  bars destined to yield from the total number of  $m - n$  bars of the main system, i.e., he is free to select one from

$$\frac{m(m-1) \dots (m-n)}{(n+1)!}$$

groups.

d) Upon obtaining the changes in length of all the principal bars, to find the deformation  $\Delta_i$  of the redundant bars. This part of calculations is of kinematic character. It is simplest to construct a Williot diagram; here it is not necessary to construct the additional part of the diagram which takes into account the rotation of the truss in order to satisfy the end conditions.

The deformation of the redundant bars can, obviously, be also determined analytically from the formula

$$\Delta_j = -\frac{X_j l_j}{EF_j} = \sum_i \frac{N_p N_j l_i}{EF_i} = \sum_i \frac{\sigma_i N_j l_i}{E}, \quad (1)$$

where  $\sigma_i$  are the specified stresses in the principal bars,  $N_j$  are forces produced in them by two equal and opposite forces  $X_j = 1$ , which replace the redundant  $j$ th bar. Calculations using this formula suffice for finding the deformation of one redundant bar; to find the deformation of all the  $n$  redundant bars, it will be necessary to repeat the calculations using the formula  $n$  times, while the Williot diagram makes it possible to obtain all these deformations from a single construction.

e) Let the graphically or analytically found deformations of the redundant bars be found to be *elastic*. In this case it is possible to pass on to the second stage of the calculation, consisting in calculating the forces  $X_j$  in these bars. The sign of these forces is known, since deformations  $\Delta_j$  have already been found.

The complication facing the designer who starts his calculations from specifying the magnitudes and signs of  $X_j$  does not exist here. As to selection of the absolute values of  $X_j$ , he is limited by the condition presented in ¶"a," i.e., one must avoid quantities so large which could change the signs of the force in any of the principal bars; for any bars one should conform to the inequality

$$\frac{N_p + \sum_{i=1}^n N_i X_i}{N_p} > 0. \quad (2)$$

It is obvious that this system of  $n - n$  inequalities can be always satisfied by selecting sufficiently small values for all the  $X_j$ ; consequently, there exists a *region* bound by some final values of these quantities. It is also inconvenient to specify too low values, since this will make the use of a statically indeterminate truss unjustified.

It is thus necessary to specify all the  $X_j$  and then to calculate the forces in the principal bars and to satisfy Inequalities (2), which should not be difficult.

f) In particular, it is possible to obtain a simple, but non-interesting solution, by equating all the  $X_j$  to zero. Since the stresses in the redundant bars, which are predetermined by the specified stresses  $\sigma_i$  of the principal bars are in general not zero, then it would follow from the condition  $X_j = 0$  that  $F_j = 0$ . Such a solution would thus require to forego designing a statically indeterminate truss and replacing it by a random statically determinate truss.

Note. The requirement expressed by Inequality (2) is not generally mandatory, i.e., one may allow such a selection of values of  $X_j$  for which the total force and stress in one or another principal bar would have a different sign, but this would then necessitate checking the translations in the directions of forces  $X_j$ ;

this check should show whether the signs of  $\lambda_j$  can be retained or they have to be changed.

g) Select the cross sections of the redundant bars from the formula

$$F_j = \left| \frac{X_j l_j}{E \Delta_j} \right|$$

and to choose the cross sections of the principal bars from the formula

$$F_i = \left| \frac{N_i + \sum N_j X_j}{\sigma_i} \right|$$

where  $\Delta_j$  are the previously found changes in length of the redundant bars;  $\sigma_i$  are the specified stresses in the principal bars.

h) It may happen that for the selected deformations of the principal bars the deformations of certain redundant bars will go past the elastic limit. Then it will be necessary to revise the selected deformations of some of the elastically operating principal bars with a view to reducing or increasing them; it may be necessary to increase the stresses in some and reduce them in some other. A graphic guide in this respect is the Williot diagram. Finally, it is possible to reclassify some of the redundant as principal and some of the principal as redundant bars. An attempt should be made to make all the deformations of the redundant bars elastic deformations.

i) Any calculation performed by the above method will result in establishing a mechanism, whose stressed-deformed state due to a given load is satisfied by equilibrium conditions of all the joints of the truss, as well as by compatibility conditions for its deformations. Here all those bars of the truss which are still in the elastic range are subjected to stresses close to the yield point. This completeness in utilizing the truss material is a feature of this method of design.

k) Check the solution. This is needed since, as the load is increased, it is possible not only for ever-new bars to start yielding, but also for some bars to return from the state of yielding to the elastic range. In the process of loading the residual deformations of bars functioning past the elastic limit produce an additional stressed state; however, the mechanism no longer experiences it. All this makes it necessary to ensure that no mechanism arises before the load reaches the specified magnitude. A recheck of a truss already designed, for which the cross sections of all the bars are already known, can be performed in the ordinary manner.

In the case when as a result of calculations the proper weight of the truss is found to differ substantially from that previously assumed, it is expedient to include it in the external load and to adjust the design accordingly.

3. As can be seen from the above presentation, the possibilities open to the designer in designing a truss on the basis of its limiting state are much more extensive than those available to him when designing the truss in the elastic range. For this reason in seeking a minimum-weight frame in the first case it has to be selected from a larger number of various trusses than in the second case. In designing for the elastic range all the bars which are a part of a given selected main system can be called equal, while when designing on the basis of the limiting state it is additionally necessary to select from these bars in one or another manner a group which is subjected to a stress equal to the yield strength.

In order to approach a minimum-weight truss it is apparently necessary to increase the stresses in the possible greatest number of bars of the main system. It would, consequently, be possible to bring the stress to the yield point not in  $n + 1$  bars, but in a greater number of bars of the principal system or even in all of its bars, if this will not involve unpermissible deformations in the redundant bars.

Such a solution, however, does not resolve the problem of a minimum-weight truss, since it is still possible to select different main systems and redundant bars, as well as to specify variously the forces  $X_j$  in the  $n$  redundant bars. We now consider the last question.

The theoretical volume of material of each bar is expressed by the formula

$$v_i = F_i l_i = \left| \frac{N_i}{\sigma_i} \right| l_i = \left| \frac{N_{iP} + \sum N_{ij} X_j}{\sigma_i} \right| l_i, \quad (3)$$

where  $N_{iP}$ ,  $N_{ij}$  are the forces in the  $i$ th bar produced in the main system by the external load and by the force  $X_j = 1$ , respectively. If the  $i$ th bar is redundant, then  $N_{iP} = 0$ ,  $N_{ij} = 0$  (when  $i \neq j$ );  $N_{ii} = 1$ .

Formula (3) is peculiar by the fact that it contains in its right-hand side the *modulo* of function  $N_{iP} + \sum N_{ij} X_j / \sigma_i$ . Methods of operating with modular functions have been developed in a number of interesting studies by Professor Yu.A. Radtsig, pertaining to the problem of designing minimum-volume trusses in the elastic range. Professor Yu.A. Radtsig has introduced for this purpose the concept of "spectral" and "out functions." For the same purpose, Professor A.I. Vinogradov has introduced the concept of the "sign variability coefficient" and of operations with functions containing this factor. Modular linear functions are also used in the theory of linear programming. The present study is done without resort to these concepts.

The partial derivative of the bar's volume with respect to variable  $X_j$  is

$$\frac{\partial v_i}{\partial X_j} = \pm \left| \frac{N_{ij} l_i}{\sigma_i} \right|. \quad (4)$$

Formula (4) is valid for the present method of design, in which the values of  $\sigma_i$  are regarded as specified. The absolute value of the right-hand side is constant, it does not depend on any of the  $X_1, X_2, \dots, X_n$ ; in general it is nonzero. It is also independent of the external load and of the total force  $N_i$  in the given bar.

As to the sign, it does depend on these quantities; it is easy to show that it is identical with the sign of the product  $N_i N_{ij} X_j$ . For example, if  $X_j$  compresses the  $j$ th redundant bar ( $X_j < 0$ ), while force  $X_i = +1$  puts bar  $i$  into tension ( $N_{ij} > 0$ ), and finally, the total force in the  $i$ th bar due to the combined external load and all the forces  $X_j$  is tensile ( $N_i > 0$ ), then the right-hand side of this formula has a minus sign.

In the method presented above the sign of  $X_j$  cannot change, since it is determined by specified and unvariable values of stresses  $\sigma_i$ . Consequently, the sign of the right-hand side can change only when the sign of force  $N_i$  changes. In this case, all the derivatives  $\frac{\partial v_i}{\partial X_j}$ , i.e., the derivatives of the volume of the  $i$ th bar with respect to any of the forces  $X_1, X_2, \dots, X_n$  change sign without changing magnitude. These derivatives are thus discontinuous functions.

The partial derivative of the total volume  $V$  of the truss is expressed by the sum

$$\frac{\partial V}{\partial X_j} = \sum_{i=1}^m \pm \left| \frac{N_{ij} l_i}{\sigma_i} \right|, \quad (5)$$

which contains all the bars of the truss. In the process of varying the magnitudes (but not signs) of forces  $X_j$ , all the terms in the right-hand side retain their absolute values without change, while the sum in the right-hand side remains either unchanged, or discontinuously changes its magnitude, or its magnitude and its sign. Discontinuities take place for such values of  $X_j$  which produce a discontinuity in any given term of the sum. Let for some given values of all the  $n$  forces  $X_j$  be found the value of the derivatives

$$\frac{\partial V}{\partial X_1}, \quad \frac{\partial V}{\partial X_2}, \quad \dots, \quad \frac{\partial V}{\partial X_n}.$$

The total differential of function  $V$

$$dV = \frac{\partial V}{\partial X_1} dX_1 + \frac{\partial V}{\partial X_2} dX_2 + \dots + \frac{\partial V}{\partial X_n} dX_n \quad (6)$$

makes it possible to point out in what direction the magnitudes of  $X_j$  should be changed in order that  $V$  would become smaller; it is

most effective to make all the terms in the right-hand side of Formula (6) negative. For this the increment of  $X_j$  in each of the  $\frac{\partial V}{\partial X_i} dX_i$  should be assigned a sign opposite to that of derivative  $\frac{\partial V}{\partial X_i}$ . This can be done with retention of the magnitudes and signs of all the stresses  $\sigma_i$ , in general, by an infinite number of methods, since all the tension bars should conform to inequalities in the form

$$N_{ip} + \sum_{i=1}^n N_{ij} X_i \geq 0,$$

while all the compression bars should obey the inequalities

$$N_{ip} + \sum_{i=1}^n N_{ij} X_i < 0$$

(a total of  $m$  inequalities containing  $n$  quantities  $X_j$ ). These inequalities are *a priori* noncontradictory, since the above method, as was pointed out previously, immediately ensures noncontradictory selection of a system of  $n$  quantities  $X_j$ .

It is obvious that the greatest effect on retention of the preselected stresses  $\sigma_i$  is obtained when  $n$  of these inequalities become equalities, i.e., for such systems of compatible values of  $X_j$ , corresponding to the limiting state, for which the truss is converted to some statically determinate truss.

Here we have an analogy with the known theorem on the weight of a truss designed for the elastic range.

The number of the possible statically determinate trusses which is formed from a given statically indeterminate truss is, as we know, very large; among them exists one, lighter than all the others, satisfying the specified stress distribution in the limiting state. This truss is still not the minimum-weight truss, since the number of possible failure mechanisms is also great, and by specifying stresses  $\sigma_i$  it is possible to apply the same deliberations to each of them. The fact that in the problem at hand one has to deal with a greater number of versions than in a similar problem of an *elastic* truss is not accidental; it is due to the fact that the failure mechanism can be selected in many ways.

Although minimum weight is obtained in a truss such for which the force distribution in the limiting state is the same as in a certain statically determinate main system, it cannot be claimed that the designing of statically indeterminate trusses is without a basis. Under operating conditions a statically indeterminate truss functions differently than a statically determinate truss, and not in the manner that it functions in the limiting state.

## §2. On Designing Minimum-Weight Structures for Dynamic Loads

1. The question of methods for designing minimum-weight structures for dynamic loads has not been as yet afforded its due attention and apparently, nothing has been published on it. The present brief remark is an extract from a much more extensive study of this problem. It has as its purpose to call attention to an important, but forgotten aspect of the problem of minimum-weight structures and is restricted to elementary examples, which, however, expose certain features which differentiate the dynamic from the static statement.

The principal difference between these two statements is of course the fact that a new independent unknown, i.e., time, appears in the dynamic problem. Due to its importance is acquired by factors which are not present in static calculations, i.e., masses and their distribution among spans, laws governing the variations in loads with respect to time. The volume of the structure's material also acquires a different meaning, i.e., while in a static study a given volume involves only modification of the load, in dynamic design it involves changes in the mass, which is related to changes in the spectrum of natural frequencies and vibratory modes of the system and, consequently, to changes in the design effect of all the loads.

As a simple example we now consider elementary design of a single-span hinged beam with a rectangular cross section constant over the entire span. It is necessary to design it for its proper weight and for an instantaneous impulsive load of intensity  $s$  kg·s/m, uniformly distributed over the entire span. The maximum normal tensile and compressive stress  $\sigma$ , as well as the maximum allowable height  $h_0$  and minimum width  $b_0$  of the cross section are given. The sought cross-sectional dimensions and its area are denoted by  $b, h, F$ .

We shall regard the beam as a system with one degree of freedom and we shall take as the natural mode of its vibration a static elastic curve corresponding to a uniformly distributed load. The natural frequency  $\omega$  is expressed by the formula

$$\omega = \frac{\pi^2}{l^3} \sqrt{\frac{EJg}{F\gamma}} = \frac{\pi^2 h}{2l^3} \sqrt{\frac{Eg}{3\gamma}} = \frac{\pi^2 h}{2l^3} v, \quad (7)$$

where  $\gamma$  is the specific weight of the beam's material; the dimensions of constant  $v$  are [m/s].

From the three possible uniformly distributed loads  $\gamma F$  and  $\gamma F \pm s\omega$  it is obvious that the design load in our case is  $\gamma F + s\omega$ . From the condition

$$M_{\max} = \frac{(\gamma F + s\omega) l^3}{8} = \frac{\sigma b h^3}{6} = \frac{\sigma F h}{6}$$

we find

$$F_d = \frac{3\pi^2 h v s}{2(4\sigma h - 3\gamma l^2)}. \quad (8)$$

When  $s = 0$ ,  $F = 0$ ; this is natural, since by the statement of the

problem the beam is intended only for taking up the impulsive load; if it is zero, the beam is not needed.

The expression for  $F$  should be positive; hence

$$h > \frac{3\gamma l^3}{4\sigma}. \quad (9)$$

As can be seen from (8), area  $F$  becomes smaller as height  $h$  is made larger; consequently, it is convenient to increase this dimension as much as possible. When  $h = \infty$

$$F = \frac{3\pi^3 v s}{8\sigma}.$$

The smallest practical minimum area is obtained when  $h = h_{\max}$

$$F_{\min} = \frac{3\pi^3 h_{\max} v s}{2(4\sigma h_{\max} - 3\gamma l^3)}. \quad (8')$$

Here

$$b = \frac{F}{h_{\max}}.$$

If it is found that  $b < b_0$ , then it will be necessary to specify  $b = b_0$  and to find from (8)

$$h = \frac{\pi^3 v s}{\sigma b_0} + \frac{2\gamma l^3}{\sigma}. \quad (10)$$

The optimality of this solution consists in the fact that for a given constant cross section it gives a beam with relative minimum weight.

2. A further weight reduction could be obtained by dropping the constant-area requirement. Let, for example, over lengths  $l/3$  of the span's extreme sections it have a constant height  $h_1$ , while at the middle section it have a height  $h_2$ . If we assume a sinusoid as the principal mode of vibrations, then we can derive the formula

$$\omega^2 = \frac{384^3 \cdot 81^3 \cdot E J_1 J_2 g}{18^3 \cdot 90 l^4 \gamma b} \times \frac{(32h_1^2 + 20h_2^2 + 50h_1 h_2) J_2 + (41h_2^2 + 30h_1^2 + 70h_1 h_2) J_1}{(0,195h_1 + 0,305h_2) [(80h_1 + 64h_2) J_2 + (141h_2 + 120h_1) J_1]^2}. \quad (11)$$

For  $h_1 = h_2 = h$ ,  $J_1 = J_2 = J$  Expression (11) becomes

$$\omega^2 = 98,3 \frac{E J g}{\gamma b h l^4}, \quad \omega = \frac{9,8}{l^2} \sqrt{\frac{E J g}{\gamma b h}};$$

in the exact solution the coefficient in this formula is 9.88.

We have two conditions for determining  $b$ ,  $h_1$ ,  $h_2$

$$\left. \begin{aligned} M\left(\frac{l}{3}\right) &= \frac{\gamma b l^3}{18} (h_1 + h_2) + \frac{s \omega l^3}{9} = \frac{\sigma b h_1^2}{6}; \\ M\left(\frac{l}{2}\right) &= \frac{\gamma b l^3}{72} (4h_1 + 5h_2) + \frac{s \omega l^3}{8} = \frac{\sigma b h_2^2}{6}. \end{aligned} \right\} \quad (12)$$

A third condition is that of minimum volume

$$V = \frac{lb}{3} (2h_1 + h_2). \quad (13)$$

This problem is solved by using Lagrange multipliers after substituting into Eqs. (12) the expressions for  $\omega$  given by Formula (11).

In the more general statement, when it is required to find the most suitable mode of variation of cross section over the span, the problem is obviously solved by means of variational calculus.

3. Let it be required to design the same beam, but of constant cross section for the proper weight and for a variable load  $P \cos \theta t$  concentrated at midspan. The design bending moment for steady-state induced vibrations of the beam, as a system with one degree of freedom, is expressed by the formula

$$M = M_{st} \pm \frac{Pl}{4 \left(1 - \frac{\theta^2}{\omega^2}\right)} = M_{st} \pm \frac{Pl}{4 - \frac{4l^2\theta^2}{\pi^2 h^3 v^2}}, \quad (14)$$

where, as before

$$v = \sqrt{\frac{Eg}{3\gamma}}, \quad M_{st} = \frac{ql^3}{8}.$$

The second term in Formula (14) has a plus sign when  $\theta < \omega$ , and a minus sign when  $\theta > \omega$ .

When  $\theta=0$  or  $\theta=\infty$ , the design bending moment becomes accordingly  $M = M_{st} + \frac{Pl}{4}$  or  $M_{st}$ , i.e., the problem becomes a static problem.

Quantities  $b$  and  $h$  are subjected to the condition

$$M_{st} \pm \frac{Pl}{4 \left(1 - \frac{l^2\theta^2}{\pi^2 v^2 h^3}\right)} - \frac{\sigma b h^3}{6} = 0 \quad (15')$$

or, in the abbreviated form

$$f(b, h) = M_{st} \pm \frac{Pl}{4 \left(1 - \frac{\alpha^2}{h^3}\right)} - \frac{\sigma b h^3}{6} = 0, \quad (15)$$

where

$$\alpha^3 = \frac{l^2\theta^2}{\pi^2 v^2}. \quad (16)$$

When conforming to Condition (15), which requires that the maximum stresses have a specified final value  $\sigma$ , resonance is impossible. In fact, for  $\frac{a^2}{h^2} = 1$  we would have obtained from (15)  $\frac{\sigma b h^2}{6} = \infty$ , i.e.,  $b = \infty$ .

The problem consists in finding the minimum of function  $F(b, h)$  on conformance to Condition (15) which relates variables  $b$  and  $h$ . Using the Lagrange multiplier we can write two equations

$$\frac{\partial F}{\partial b} - \lambda \frac{\partial f}{\partial b} = 0 \text{ and } \frac{\partial F}{\partial h} - \lambda \frac{\partial f}{\partial h} = 0.$$

where  $\lambda$  is constant. After this factor is eliminated and expressions for  $F$  and  $f$  are substituted, we can write the equation

$$\mp \frac{Pl a^2}{2 \left(1 - \frac{a^2}{h^2}\right)^2 h^2} - \frac{\sigma b h}{6} = 0, \quad (17)$$

which should be solved simultaneously with (15).

In the case when  $h > a$ , i.e.  $\omega > 0$ , the first term of Eq. (17) has a minus sign; consequently, it can be satisfied only on the condition that one of the variables  $b$  or  $h$  are negative. This shows that there is no minimum. In fact, when  $\omega > 0$  it is expedient to increase the rigidity of the beam, increasing height  $h$  without bounds, since this brings the dynamic effect of force  $P \cos \theta t$  closer to the static effect.

When  $\omega < 0$  the first term in Eq. (17) should be taken with the plus sign and then this equation should be solved simultaneously with Eq. (15). Elimination of parameter  $b$  yields the equation

$$\left(M_{st} - \frac{Pl}{4}\right) h^4 - \left(2M_{st} + \frac{Pl}{4}\right) a^2 h^2 + M_{st} a^4 = 0, \quad (18)$$

from which will be found two positive values of  $h^2$ . The value of  $b$  will be found from the corresponding equation (17). One of these two pairs of values of  $b$  and  $h$  will be the solution of the minimum-volume problem.

As a result of the fact that the solution satisfies the inequality  $\omega < 0$ , the steady-state vibrations produced by force  $P \cos \theta t$  will take place after the beam will have passed through resonance. Hence the dimensions  $b$  and  $h$  thus found should be checked for beam strength in this transient regime.

4. A single-spar hinged beam with constant rectangular cross section is loaded by an instantaneous uniformly distributed impulsive load of strength  $s$  kg·s/m. It is required to select dimensions  $b$  and  $h$  so as to obtain minimum weight in the limiting state (with respect to the carrying capacity).

In the elastic range the deflections and the bending moment are expressed by the formulas

$$M(x, t) = \frac{4s}{\pi} \sqrt{\frac{EJ}{\mu}} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l} \sin \omega_n t; \quad (19)$$

$$y(x, t) = \frac{4sl^2}{\pi^3 \sqrt{EJ\mu}} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{l} \sin \omega_n t, \quad (20)$$

where

$$\mu = \frac{F\gamma}{g}.$$

From the equation of moments it is possible to determine values of  $x$  and  $t$  corresponding to the first appearance of the plastic hinge if it is assumed that up to this moment the beam functions in the elastic range. Let this take place at some time  $t_0$  and in a cross section with abscissa  $x_0 = l/2$ . At this instant the beam will have the kinetic energy

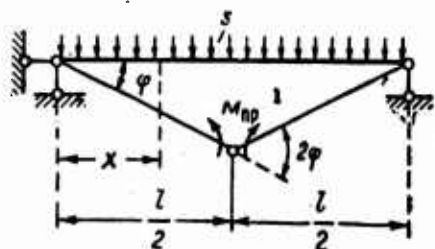


Fig. 2

$$K_0 = \frac{\mu}{2} \int_0^l \left[ \frac{\partial}{\partial t} y(x, t) \right]^2 dx$$

and potential energy

$$\Pi_0 = \frac{EJ}{2} \int_0^l \left[ \frac{\partial^2}{\partial x^2} y(x, t) \right]^2 dx.$$

The further motion can be studied by assuming that at the instant when the kinematic chain (Fig. 2) reaches its extreme bottom position the entire potential and kinetic energy are absorbed by the work of the plastic moment  $M_{pr}$

$$\Pi_0 + K_0 = M_{np} \cdot 2\varphi_{max}. \quad (21)$$

from this it is possible to express  $\varphi_{max}$  in terms of parameters  $b$  and  $h$ .

The limiting value of  $\varphi_{max}$  should be specified; consequently, quantities  $b$  and  $h$  in Eq. (21) should be regarded as unknown. Thus, the problem is again reduced to finding the minimum of function  $F = bh$  in the presence of an equation relating the two parameters.

The examination of these several examples should, obviously, be regarded as only an introduction to the given problem.

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# A STUDY OF THE COST FUNCTION IN THE DESIGN OF OPTICAL SYSTEMS

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(Khar'kov)

The use of electronic digital computers [EDC] (ЭЦМ) in structural mechanics imparts particular interest to the study of problems of design of optimal systems and developing algorithms for determining optical forces in bar systems. Up to present extensive use is not made of methods for designing optical systems in design practice. This is due in part to the complexity of such problems, cumbersomeness of calculations, which are too inconvenient for manual computation. The finding of an optical system is basically a problem of structural synthesis. Here may arise problems which are entirely new for structural mechanics, such as does an optimal force distribution exist in the given class of systems? Is it the only solution, is it possible to have other designs close in weight to the optimal, but differing from it by the magnitudes of forces in the cross sections? Does the volume or cost function have such common properties which allow for constructing a process of successive approximations, convenient for computer calculations and suitable for all problems of practical importance.

This article examines certain general theorems on the optimal force distribution in an ensemble of structures with specified outline of axes, for different optimality criteria. The concept of an ensemble of structures with specified outline of axes, introduced earlier, is assumed to be known [9].

## §1. Concerning Optimality Criteria of a Bar System with Flexure Predominant

In the ensemble of structures with a specified outline of axes we consider the functional

$$V = \int_L |M(s)|^\alpha u(s) ds, \quad (1)$$

where the integration extends over the entire outline  $L$  of the axes of a system with flexure predominant,  $M(s)$  is the value of the design-moment function in section  $s$ ,  $u(s) > 0$  is the principal or weight function, and  $\alpha > 0$ .

Usually the problem of minimum theoretical volume or minimum cost of a system can be reduced to finding the minimum of Functional (1). Let us consider certain particular cases.

1. For the known strength condition

$$\sigma = \frac{|M|}{W}, \quad (2)$$

where  $\sigma$  is the positive value of the specified stress, we introduce the core radius of the cross section  $w = \frac{W}{F}$ ,  $F$  being the cross-sectional area. We get  $u(s) = \frac{1}{\sigma W}$ ,  $\alpha = 1$ . The functional is the theoretical volume of a system with variable-stiffness elements.

2. In designing for the ultimate equilibrium we set  $\sigma_r = \frac{|M|}{T}$ , where  $T$  is the plastic section modulus,  $w' = \frac{T}{F}$ . Then  $u(s) = \frac{1}{\sigma_r W'}$ ,  $\alpha = 1$ , and Integral (1) is the theoretical volume of a system in which the design moments do not exceed the limiting values.

3. As related to the principal function  $u(s) > 0$  the functional has no minimum; its exact lower boundary is zero and is attained when  $u(s) = 0$ . In terms of prerequisites this is due to lack of consideration of tangential stresses in the strength conditions. Hence the values of  $u(s)$  have to be specified on structural considerations on the basis of trial designs; this problem has already been considered.

In attempting to eliminate the "principal function" it is possible to impose explicit design limitations on the form of cross sections. Let for a boxlike or wide-flange symmetrical cross section  $h$  be the cross-section height,  $b$  the flange width,  $d$  the total web thickness,  $t$  the flange thickness,  $t = kd$  and  $b = \beta h$ . Approximately, we set

$$F = dh + 2tb, \quad W = \frac{dh^3}{6} + bth.$$

Then for strength condition (2), when  $d = \text{const}$ ,  $t = \text{const}$  and  $\beta = \text{const}$ , we get the theoretical volume of the system

$$V = \sqrt{\frac{6(1+2k\beta)^3}{1+6k\beta}} \cdot \frac{d}{\sigma} \int_L |M(s)|^{\frac{1}{2}} ds. \quad (3)$$

In this case,  $u(s) = \text{const}$ ,  $\alpha = 1/2$ . We note that when  $6k\beta = 1$  we have an analytical minimum of cross-sectional areas for a given web and flange thickness. On the basis of the minimum analytic area in a statically determinate system it is necessary to simultaneously vary both the cross-sectional width and height, maintaining their ratio constant, if the flange and web thicknesses remain constant.

4. However, simultaneous changing of the cross-sectional height and width may be found unconstructive. Let for a boxlike or wide-flange section the web height and thickness be specified, and the flanges be selected on the basis of strength criteria. Then the theoretical volume of the structure will approximately be

$$V = \int_L \frac{hd}{3} ds + \int_L \frac{2}{\sigma h} |M(s)| ds, \quad (4)$$

i.e., it differs by a constant from the function in Form (1), in which we should set  $u(s) = \frac{2}{\sigma h}$ ,  $\alpha = 1$ .

5. Finally, we assume that the cost of a design execution of a cross section is directly related to the magnitude of the force; in this case  $u(s)$  denotes the cost per unit length and per unit absolute magnitude of the moment; the functional denotes the total cost of the system or the cost to within an initial constant.

For a single external load and  $\alpha = 1$  the problem simplifies and in particular cases it reduces to the problem of the theory of approximation of functions which has been already considered by P.L. Chebyshev [1], Ye.I. Zolotarev [2] and subsequently by others [3-5].

The problem of structural mechanics on minimum volume of a system with  $\alpha = 1$  was considered without reference to the above mathematical studies and has an extensive literature; the reference list in the survey article by Z. Wasiutynsky and A. Brandt lists 234 references [7].

Of importance was the monograph by I.M. Rabinovich, devoted to the theory of statically indeterminate trusses [6].

The problem of minimum cost for a single external load and  $\mu(s) = 1$  was also examined previously; [8] is a study of some properties of the cost function for this case.

It follows from this that, for different optimality criteria, the obtaining of an optimal system reduces to finding such a force distribution for which a functional such as (1) would be at minimum. The most interesting case is that when the system is loaded by different momentary loads. We shall examine this general case, dividing the loads into a constant and momentary, independent and incompatible. This classification was previously derived by this author in [10]. We introduce the concept of a design moment; the results are easily extensible to the case of longitudinal design forces, which is important for trusses.

## §2. Function of the Design Moment

1. Let the outline of the axes of a plane bar system with a total number  $n$  of redundant junctions be specified as piecewise smooth curves; parameter  $s$  is used for determining the position of a point on the axes in the usually assumed sequence of contour traverse. We specify some value  $s = s_0$  and consider the forces in a given cross section of a system with flexure predominant. Let:

- $M_0$  be the moment due to the constant load;
- $M_k$  be the moments due to independent loads ( $k=1, 2, \dots, l$ ) and moments due to incompatible loads ( $k=l+1, \dots, m$ ).

Let  $M_A$  be the sum of moments due to a given "ensemble" of loads, i.e.,  $M_A = M_0 + M_{.1} + M_{.2} + M_{.k}$ .

We denote:  $M_k \in M_A$  — moment  $M_k$  is a part of sum  $M_A$  as a com-

ponent;  $M_k \notin M_A$  — moment  $M_k$  is not a component of sum  $M_A$ . For a linearly deformable system we establish rules of composing sums of moments:

1) always  $M_0 \in M_A$ , the moment due to the constant load is a component of any sum of moments;

2) for  $k=1, 2, \dots, t$   $M_k \in M_A \iff M_k \in M_A$  irrespective of the moments and other loads;

3) for  $k=t+1, \dots, m; r=t+1, \dots, m; k \neq r$ ; if  $M_k \in M_A$ , then always  $M_r \in M_A$

It is obvious that these rules reflect the real conditions of combinations of independent and incompatible loads in their effect on a structure.

Let, for example,  $M_0$  be the moment due to the constant load,  $M_1$  and  $M_2$  be the moments due to independent loads, and  $M_3$  and  $M_4$  be the loads due to incompatible loads, the possible sums then are:  $M_0, M_0+M_1, M_0+M_2, M_0+M_1+M_2, M_0+M_3, M_0+M_1+M_3, M_0+M_2+M_3, M_0+M_1+M_2+M_3, M_0+M_4, M_0+M_1+M_4, M_0+M_2+M_4, M_0+M_1+M_2+M_4$ .

By definition we shall call sum  $M_A$  with a given ensemble of terms the *design moment* and denote its magnitude by  $M$ , if

$$|M_A| \geq |M_s|, \quad (5)$$

where  $M_s$  are all the possible sums composed according to Rules 1-3 and different from  $M_A$  by the ensemble of terms.

It is clear that when  $M = 0$  all the  $M_k = 0$  ( $k=0, 1, \dots, m$ ).

In a system of a set of designs with a specified outline of axes the bending moment in section  $s$  due to each of these loads is expressed in the well-known manner

$$M_k(s) = M_p^k(s) + \sum_{i=1}^n \bar{M}_i(s) X_i^k, \quad (6)$$

where  $X_i^k$  is the value of the  $i$ th unknown in the principal system of the method of forces due to the  $k$ th load,  $M_p^k(s)$  is the moment in the  $s$ th section due to the  $k$ th load in the principal system,  $\bar{M}_i(s)$  is the moment in the  $s$ th section due to the  $i$ th unit unknown,  $\bar{X}_i = 1$ ,  $n$  is the total number of redundant couplings.

Variables  $X_i^k$  will be regarded as the independent variables of the problem. The ensemble of values  $X_i^k = a_i^k$ ,  $i=1, 2, \dots, n$ ,  $k=0, 1, \dots, m$  will be regarded as a point in an  $(m+1)n$ -dimensional space of redundant unknowns, which is denoted by  $X$ . Then Functional (1) can be regarded as a function of  $(m+1)n$  variables in the space of redundant unknowns for a given outline of axes. The set of structures with specified outlines of axes is a mechanical interpreta-

tion of the domain of distribution of the function of the design moment

$$\{-\infty < X_i^k < +\infty; i = 1, 2, \dots, n; k = 0, 1, 2, \dots, m; s \in L\},$$

which is denoted by  $(X, L)$ .

By definition, this set contains all the statically indeterminate systems with a total number of redundant couplings and with partially removed couplings, all the statically determinate systems and mechanisms which are in equilibrium, and the existence of initial forces for any kind of loading is permitted [10]. The mechanical meaning of the last assumption can be explained as follows: let the redundant couplings of the system be provided with devices, which for any load may impart to the system any preset values of forces in these couplings. Then the forces in the redundant couplings for any load can be regarded as independent, not bound by compatibility conditions for deformations. In general, optimal forces will belong to different systems in the set. However, the remarkable property of this problem consists in the fact that for certain classes of load and of outlines of axes the optimal forces are achieved in a single system.

Forgoing the usual representation of a computational system and replacing it by the more complex concept of a set of structures unexpectedly results not only in a correct statement of a problem which has a unique solution in the given set under some conditions, but also in a simpler solution of the problem, which at the same time has a quite general meaning.

Let us consider some properties of the function of the design moment, which are needed for the subsequent discussion.

2. Let  $\alpha$  and  $\beta$  be points in the space of redundant unknowns. The corresponding values of  $X_i^k$  due to each of the loads will be denoted by second subscripts  $\alpha$  and  $\beta$ . We select point  $\gamma$  on the segment  $(\alpha, \beta)$ , setting

$$X_{i\gamma}^k = (1 - \nu) X_{i\alpha}^k + \nu X_{i\beta}^k, \quad (7)$$

where

$$0 < \nu < 1.$$

Then for each of the loads in the specified section  $s = s_0$  (the notation which we shall drop for brevity) we have

$$M_{k\alpha} = M_p^k + \sum_{i=1}^n \overline{M}_i X_{i\alpha}^k;$$

$$M_{k\beta} = M_p^k + \sum_{i=1}^n \overline{M}_i X_{i\beta}^k;$$

$$M_{k\gamma} = M_p^k + \sum_{i=1}^n \overline{M}_i X_{i\gamma}^k.$$

It follows from (7) that

$$M_{k\gamma} = (1 - \nu) M_{k\alpha} + \nu M_{k\beta}. \quad (8)$$

It is obvious that this property is retained also for the sum of moments. Hence we shall denote by A, B, Γ the design combinations of loads for points α, β, γ, respectively. The moments from these combinations of loads will be denoted by subscripts A, B, Γ. By definition of the design moment

$$\begin{aligned} |M_A^a| &\geq |M_\Gamma^a|; & |M_B^b| &\geq |M_\Gamma^b|; & |M_\Gamma^c| &\geq |M_A^c| \\ |M_A^a| &\geq |M_B^b|; & |M_B^b| &\geq |M_A^a|; & |M_\Gamma^c| &\geq |M_B^b|. \end{aligned}$$

By virtue of linearity it follows from (8)

$$M_\Gamma^c = (1-v)M_\Gamma^a + vM_\Gamma^b,$$

whence

$$\begin{aligned} |M_\Gamma^c| &= |(1-v)M_\Gamma^a + vM_\Gamma^b| \leq (1-v)|M_\Gamma^a| + v|M_\Gamma^b| < \\ &< (1-v)|M_A^a| + v|M_B^b|. \end{aligned} \quad (9)$$

We prove the auxiliary inequality

$$[(1-v)|M_\Gamma^a| + v|M_\Gamma^b|]^k \leq (1-v)|M_\Gamma^a|^k + v|M_\Gamma^b|^k, \quad (10)$$

where  $k \geq 1$ . In cases of  $k=1$ ;  $v=0$ ;  $v=1$ ;  $M_\Gamma^a = 0$ ;  $M_\Gamma^b = 0$  Condition (10) is trivially satisfied. For  $k > 1$ ,  $0 < v < 1$  and nonzero moments we use Inequality (C), which is easy to get.

We set

$$f(v) = \frac{1}{(1-v)^{k-1}} + \frac{x^k}{v^{k-1}}; \quad 0 < v < 1.$$

In the interval  $0 < v < 1$  for a fixed  $x > 0$  this is a continuous differentiable function which has an analytical minimum for  $v = \frac{x}{x+1}$ , which is equal to  $(1+x)^k$ , here if  $v \rightarrow 0$  or  $v \rightarrow 1$ , then  $f(v) \rightarrow +\infty$ .

From this follows the inequality

$$(1+x)^k \leq \frac{1}{(1-v)^{k-1}} + \frac{x^k}{v^{k-1}}; \quad k \geq 1. \quad (C)$$

Setting in it

$$x = \frac{|M_\Gamma^b| v}{|M_\Gamma^a| (1-v)},$$

we get (10).

From (10) and from the definition of design moments we get the important inequality for design moments in points (α, β, γ for  $k \geq 1$ ,  $0 \leq v \leq 1$ .

$$|M_\Gamma^c|^k \leq (1-v)|M_A^a|^k + v|M_B^b|^k. \quad (11)$$

Since (11) is valid for any  $s_0 \in L$ , then this thus proves the property of downward convexity of the module of the design moment in the space of redundant unknowns for any point on the axes of a specified outline.

Inequality (11) allows us to establish the convexity property for (1) when  $\alpha \geq 1$ . However, for  $0 < k < 1$  auxiliary inequality (C) changes sign

$$(1+x)^k \geq (1-v)^{1-k} + v^{1-k} x^k, \quad 0 < v < 1.$$

Substitution

$$x = \frac{|M_F^3|v}{|M_F^2|(1-v)}$$

yields, instead of (10), the opposite inequality

$$\begin{aligned} [(1-v)|M_F^2| + v|M_F^3|]^k &\geq (1-v)|M_F^2|^k + v|M_F^3|^k; \\ 0 < k < 1, \end{aligned} \quad (12)$$

and since the modules of the moments in the right-hand side only do not exceed the modules of moments in points  $\alpha$  and  $\beta$ , then Condition (11) becomes invalid for  $0 < k < 1$ .

3. Let  $M_p^k(s), \bar{M}_i(s)$  be functions piecewise continuous in  $s$ . We consider the point of continuity of these functions  $s_0 \in L$ . Since the linear combinations of continuous functions and their moduli are continuous, then for a specified  $\alpha \in X$  we select  $\delta$  in a manner such as to for  $\epsilon > 0$  satisfy for all the sums set up according to Rules II-III the conditions

$$|M_R(s)| - |M_R(s_0)| < \epsilon, \quad |s_0 - s| < \delta.$$

This condition is satisfied for all the sums, and thus also for sum  $M_A$  which is the design sum in  $s_0$ .

If sum  $M_A$  remains the design sum over the entire interval  $(s_0 - \delta, s_0 + \delta)$ , then its modulus is a continuous function. If, however, in point  $s_1$ ,  $|s_1 - s_0| < \delta$ , we have  $|M_B(s_1)| > |M_A(s_1)|$ , then, anyway we will have the following:

for  $|M_A(s_0)| > |M_B(s_1)|$

$$\begin{aligned} ||M(s_0) - M(s_1)|| &= ||M_A(s_0) - M_B(s_1)|| < ||M_A(s_0)| - \\ &- |M_A(s_1)|| < \epsilon; \end{aligned}$$

for  $|M_A(s_0)| < |M_B(s_1)|$

$$\begin{aligned} ||M(s_0) - M(s_1)|| &= ||M_A(s_0) - M_B(s_1)|| < \\ &< ||M_B(s_0) - M_B(s_1)|| < \epsilon. \end{aligned}$$

It was thus proven that the modulus of the design moment is a continuous function in the continuity points  $M_p^k(s), \bar{M}_i(s)$  ( $k=0, 1, 2, \dots, m, i=1, 2, \dots, n$ ).

With a view to applications we shall henceforth assume without special statements that functions  $M_p^k(s)$  are bound and piecewise-continuous with a finite number of first-order points of dis-

continuity, while functions  $M_i(s)$  will be regarded as continuous.

4. We note that Condition (5) can be satisfied for sums in the left-hand side with different ensembles of terms, which then will all be design sums. In particular cases these will be sums with different signs. We shall by convention assume that for a given section sums which include or do not include the moment due to a particular load which is equal to zero as different with respect to the ensemble of terms, even if the magnitudes of these sums are identical. In precisely the same manner the sign of the design moment in the given point  $s_0 \in L$  will be thought to be the sign of a definite, somehow marked design sum from among different design sums, retaining in the vicinity of point  $s_0$  the mark on this sum with the same ensemble of terms, if the marked sum remains the design sum in this vicinity.

The need for the above provision is clarified further down and is due to the use of signum functions of the design moment  $M$  in the subsequent presentation.

5. Let us see how the main inequality (5) is satisfied for a specified  $s_0 \in L$  in different points of space of redundant unknowns  $X \{ -\infty < X_i^k < \infty ; i=1, 2, \dots, n; k=0, 1, 2, \dots, m \}$ .

If in point  $a \in X$  we have a rigorous inequality for one of the sums  $M_A$

$$|M_A| > |M_s|, \quad (13)$$

where  $M_A$  is by definition the design sum and  $M_s$  are all the possible sums set up according to Rules I-III and differing from  $M_A$  by the set of their component terms, then obviously

$$M_A + M_s \neq 0,$$

but the opposite is not true.

If  $M_k = 0$  for at least one of the independent loads ( $k = 1, 2, \dots, t$ ), then the inclusion of this term into the design sum or the lack of such an inclusion does not affect the magnitude of the sum; consequently, Inequality (13) is not satisfied.

If for one of the incompatible loads  $M_k = 0$  ( $k = t+1, \dots, m$ ) and  $M_k \in M_A$ , then Condition (13) is also not satisfied.

Consequently, if (13) is satisfied, then  $M_k \neq 0$  for all the independent loads and  $M_k \neq 0$  for incompatible loads which are components of the design sum. On Condition (13) the design sum is unique. It is clear that then  $M \neq 0$ .

Let Condition (13) be satisfied for  $M_A$ ; then it will be possible to find an  $\epsilon > 0$  such that

$$|M_A| > |M_s| + \epsilon.$$

Let us consider one of the terms  $M_k \in M_A$  ( $k \neq 0$ ), let variable  $X$  acquire an increment  $\Delta X_i^k$ , then for  $M_i \neq 0$  we get from (6) the increment in the module of the sum

$$\Delta |M_A| = \text{sign } M_A \bar{M}_i \Delta X_i^k.$$

We set

$$|\Delta X_i^k| = \frac{\varepsilon}{2 |\bar{M}_i|}.$$

Then

$$|M_A| + \Delta |M_A| \geq |M_A| - |\bar{M}_i| \cdot |\Delta X_i^k| = |M_A| - \frac{\varepsilon}{2} > |M_B|.$$

On the other hand, if  $M_k \in M_B$ , where  $M_B$  is any from the sum  $M_B$ , then

$$\Delta |M_B| = \text{sign } M_B \bar{M}_i \Delta X_i^k;$$

$$|M_B| + \Delta |M_B| \leq |M_B| + |\bar{M}_i| \cdot |\Delta X_i^k| = |M_B| + \frac{\varepsilon}{2}$$

and from this

$$|M_A| + \Delta |M_A| \geq |M_A| - \frac{\varepsilon}{2} > |M_B| + \frac{\varepsilon}{2} \geq |M_B| + \Delta |M_B|,$$

and, consequently, the load combination A remains the design combination in some vicinity of point  $\alpha$ . From this, if  $M_k \in M_A$  and (13) is satisfied, then the function of the design moment in this point has the ordinary derivative

$$\frac{\partial |M|}{\partial X_i^k} = \frac{\partial |M_A|}{\partial X_i^k} = \bar{M}_i \text{sign } M_A.$$

It is also easy to prove that on Condition (13)  $M_k \in \bar{M}$ , then also in some vicinity of point  $\alpha$ , where this condition is satisfied  $M_k \in M$ , so that in this case

$$\frac{\partial |M|}{\partial X_i^k} = 0.$$

The ensemble of points  $\alpha \in X$ , where (13) is satisfied is an open set.

If Condition (13) is not satisfied, then in the given point  $\alpha \in X$  there exist in general only one-sided partial derivatives of the moduli of the design moment.

### §3. On the Convexity of the Cost Function

1. In the most general case (see §1) Integral (1) expresses the theoretical cost of a bar system; hence for brevity we shall call it the cost function. As we have seen, this function is defined in the space of redundant unknowns for a specified outline  $L$  of the axes.

Let  $\alpha \in X$ ; we shall denote the function of the design moment for  $s \in L$  by  $M(s, \alpha)$  and the value of the cost function in the point will be denoted

$$V(\alpha) = \int_L |M(s, \alpha)|^p u(s) ds; \quad c > u(s) > 0, \quad (14)$$

where function  $M(s, \alpha)$  is set up according to Rules I-III:

$$M(s, \alpha) = M_A(s, \alpha),$$

if

$$|M_A(s, \alpha)| \geq |M_s(s, \alpha)|,$$

where  $M_s(s, \alpha)$  are sums composed according to Rules I-III and differing from  $M_A(s, \alpha)$  by the set of component terms.

In selecting the sum  $M(s, \alpha)$  in different points  $s \in L$  we have reference to the stipulation of ¶4 of the preceding section, i.e., if in this point we take the design sum

$$M(s, \alpha) = M_A(s, \alpha) \quad (15)$$

as a function of the design moment and this sum remains the design sum in the vicinity of point  $s$  for a given  $\alpha$ , then Condition (15) remains valid also in this vicinity.

For  $\delta \geq 1$  we have the following property of downward convexity of the cost function in the space of redundant unknowns.

If we have given a point  $\gamma \in X$  on the segment  $(\alpha, \beta)$  in the space of redundant unknowns

$$X_{i\gamma}^k = (1 - v)X_{i\alpha}^k + vX_{i\beta}^k,$$

where

$$\alpha \in X, \beta \in X, 0 < v < 1,$$

then for  $\delta \geq 1$

$$V(\gamma) \leq (1 - v)V(\alpha) + vV(\beta). \quad (16)$$

To prove this we use Inequality (11), i.e., when  $\delta \geq 1, 0 < v < 1$  we have

$$|M_\gamma|^{\delta} \leq (1 - v)|M_\alpha|^{\delta} + v|M_\beta|^{\delta}.$$

Since this inequality is valid for the design moments in each point  $s \in L$  and the function of the design moment has only a finite number of first-kind discontinuities, then, multiplying both parts of (11) by the principal function  $u(s) > 0$  and integrating over the specified outline  $L$  of the system's axes, we get

$$\int_L |M(s, \gamma)|^{\delta} u(s) ds \leq (1 - v) \int_L |M(s, \alpha)|^{\delta} ds + v \int_L |M(s, \beta)|^{\delta} ds,$$

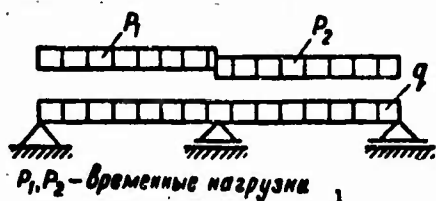
i.e., Property (16).

From this it follows that the cost function in the space of redundant unknowns does not have a maximum, and it either has a minimum in some singular point  $x_0 \in X$ , which may even not be the analytical minimum, or takes on some constant minimum value in a certain closed domain of the space of redundant unknowns.

It is obvious that the above convexity property is of substantial importance in constructing an algorithm of successive approximation of the optimum force distribution in practical design of optimal bar systems.

This is, apparently, the first proof for the general case of independent and incompatible momentary loads.

2. It is interesting that the convexity property does not apply to the cost function for  $0 < \delta < 1$ . In this case, Inequality (11) is not satisfied, it is replaced by the reverse inequality (12). No conclusion can be made with respect to the moduli by the design moment, without making additional assumptions about the properties of functions  $M_p(s)$ ,  $M_i(s)$  and the properties of the curves according to which the axes of the system are drawn. Hence it was found that even the relatively nonrestrictive requirement that if  $V(\alpha) = C$ ,  $V(\beta) = C$ , then we do not always have  $V(\gamma) < C$  was also found not valid. The cost function for  $0 < \delta < 1$  can thus several times take on a relatively smallest value in the space of redundant unknowns and even have a maximum. Here is an elementary example for the scheme of Fig. 1



where

$$V = \int_0^1 |M|'' dx,$$

$$M_p = \begin{cases} 0, & 0 \leq x < 0.5 \\ -1, & 0.5 \leq x \leq 1; \end{cases}$$

$$M = M_p + X.$$

Fig. 1. 1) Momentary loads.

In this case  $V(0) = 0.5$ ;  $V(0.5) = \frac{\sqrt{2}}{2}$  (maximum) and  $V(1) = 0.5$ . We note that if we take  $\delta = 1$ , then the cost function will remain constant in the interval  $(0, 1)$ . This points to the importance of the optimality criterion in the reverse problem; in conclusion we examine examples.

In the particular case of one load the properties of the cost function were examined in [8].

#### §4. Illustrative Examples. Results

The preceding deliberations were fully rigorous for a multi-dimensional system, and we did not dwell on examples in order to retain the continuity of presentation. Let us now consider graphic illustrations of the general theorems for the one-dimensional case, which will help in clarifying their meaning. These are elementary examples, but in practice, when designing with the help of EDC, one may encounter complex problems, and the knowledge of the properties of the problem is needed for working out algorithms for these calculations.

1. The meaning of differential properties of the module of the design moment is clarified by an example of a beam in Fig. 2.

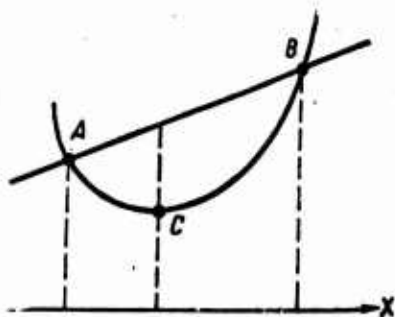


Fig. 2

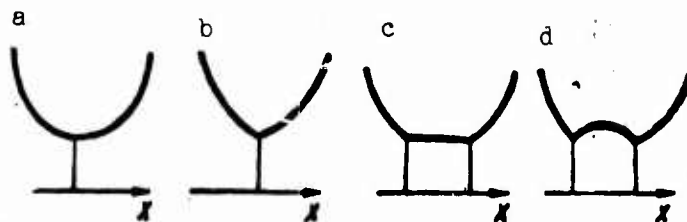


Fig. 3

For a momentary load in the left span it is convenient to have a support moment of  $X = 0$  (this, as is known, makes expedient prestressing the structure [14]). However, then the cost function does not have an analytic minimum, i.e., it reaches its lowest value in the point where the ordinary derivative does not exist, and it only changes sign. It is possible to find and use one-sided derivatives for determining the most expedient distribution of forces [15]. The graph of the cost function has the form shown in

Fig. 3b. We note that to use the ordinary procedures it is possible to use here the method of adding a vanishingly small load, which was suggested by this author previously.

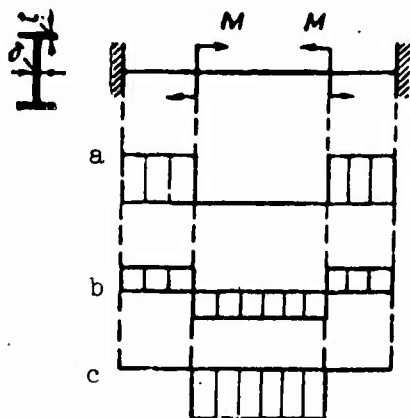


Fig. 4

2. Of particular importance is the convexity property, which was proven above for the general case of any momentary and constant loads, if the optimality criterion satisfies the condition  $\delta \geq 1$  in Formula (14). If the cost function is convex, it is possible to use methods of mathematical programming for developing cost minimization properties on EDC [11], [13]; an examination of these methods is not included here. Hence it is of great importance not only to the programmer,

but also to the designer who formulates the optimality criterion for finding the most expedient system, to have a clear idea whether the above property can be ensured. We have seen that it does not always take place and is by far not obvious, as would appear at first sight. The convexity property is clarified by the drawing of Fig. 4. If the function is convex, then for any two points A, B at its boundary the graph of the function between these points will not pass higher than the straight line connecting them. Consequently, cases are possible which are shown in Fig. 4a, b, c. But the case of Fig. 4d is impossible for a convex function.

It is actually not important which of the first three cases takes place. However, if there is no convexity, as in the case of Fig. 4d, we can no longer conduct an algorithm which would produce a single minimum cost. We recall that our illustration on the

drawing pertains to the simplest case, while in the multidimensional case it is possible to have unpleasant surprises which could not be easily explained without reference to the theory of the problem. In general, the need for developing algorithms for designing optimal systems on EDC imparts particular importance to problems of theory, which require consideration. Graphic considerations alone do not suffice for complex problems.

3. For correct presentation of the substance of the matter it is also important to clarify that the optimality criteria will not necessarily be specified in the analytic form, and the cost function proper to be minimized may not even be written in algebraic form. For example, we may choose in each step of the successive approximation specific cross sections of the most convenient shape with certain design limitations, taking into account the web stability, convenience in making the box shape, minimum flange thicknesses, etc. The most expedient values of the redundant unknowns may also be found by the successive approximation algorithm or by some other multistep method. This process will be *a priori* converging to a singular solution only in the case when the optimality criterion will ensure convexity of the cost function.

4. It is remarkable that this process may be not converging to a unique solution in the following important case: when we select at each step wide-flange or box-shaped cross sections with the most expedient ratio of dimensions, for the given minimum web and flange thickness. Since in this criterion the power index in Formula (1) is  $\frac{1}{2}$ , one cannot guarantee uniqueness of solution. In fact, for a discontinuous moment diagram the cost of a system on varying a redundant unknown may first decrease, then increase, then decrease again, etc. Figure 1 shows an example of such an elementary problem consisting of two concentrated moments acting on a beam. It turns out that equalizing the support and span moments in such a beam with equal-strength sections of the most expedient shape yields an appreciable excess of the theoretical cost over the absolute minimum, which is reached in two "ultimate" beams (diagrams in Fig. 1a, c). Of course, in this example the optimal beams are only theoretical solutions, but by prestressing the system it is possible to obtain any moment distribution. This problem is not important in itself, but rather the possibility of occurrence of such cases in the practice of programing the design of optimal systems.

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## REGIONS OF FEASIBLE FORCES IN STATICALLY INDETERMINATE TRUSSES, SUBJECTED TO NONSIMULTANEOUS LOADS

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In designing statically indeterminate trusses by the method of specified stresses [1, 2] it is necessary to specify forces in the redundant bars. In order not to obtain in this process unfeasible results (negative bar cross sections), it is necessary to specify feasible values of the redundant unknowns.

The present author showed [3] that in the case of a statically indeterminate truss loaded by a single load, the regions of feasible forces are closed parts of an  $n$ -dimensional space, separated out in by  $(n - 1)$ -dimensional "planes"

$$N_i = a_{i1} X_1 + a_{i2} X_2 + \dots + a_{ir} X_r + \dots + a_{in} X_n + b_i = 0.$$
$$i = 1, 2, 3, \dots, m.$$

Here  $X_r$  is the force in the  $r$ th redundant coupling;  $a_{ir}$  is the force in the  $i$ th bar of the statically determinate principal system produced by force  $X_r=1$ ;  $b_i$  is the same as above, due to a specified load;  $N_i$  is the force in the  $i$ th bar of a statically indeterminate truss due to a specified load;  $m$  is the number of all the elastic bars of the truss (not including the absolutely necessary bars);  $n$  is the number of redundant couplings. Later K.M. Khuberyan [4] has arrived at similar conclusions for the case of a single load.

In designing a truss subjected to a number of nonsimultaneous loads by the method of specified stresses it is also necessary to specify redundant unknowns due to each of the loads [5, 6, 7]. It is therefore natural to clarify what are the regions of feasible forces in this case and to provide a method for finding them. The functioning of a truss under a constant and momentary loads in various combinations of the latter, obviously, reduces to the preceding case, since the design combinations of these loads do not act simultaneously on the design.

We start the consideration of the problem from elementary cases.

## §1. A Once Statically Indeterminate Truss Subjected to Two Loads

1. Figure 1 shows a truss with two loads applied to it alternately. We now construct regions of feasible forces for each of these loads. We shall assume that bar 1 is redundant, cut it, and in the primary system thus obtained we shall find forces produced by the unit redundant unknown and by the loads. In order not to introduce new notation for the action of several loads, we shall provide the assumed notations for a single load by a second subscript, which shows the number of the corresponding load. Thus, for example,  $X_{r2}$  denotes the  $r$ th redundant unknown due to the second load,  $b_{ik}$  denotes the force in the  $i$ th bar of the primary system, while  $N_{ik}$  denotes the force in the same bar of the specified statically indeterminate truss due to the  $k$ th load, etc. The meaning of the subscripts of  $a_{ir}$  obviously remains unchanged, since this force is not produced by a load, but from the unit force of the redundant unknown.

Forces  $a_{i1}$ ,  $b_{i1}$ ,  $b_{i2}$  in the assumed primary system are shown in Table 1.

TABLE 1

№ стержней A	$a_{i1}$	$b_{i1}$	$b_{i2}$
1	1	0	0
2	-1	$1,5P$	$P$
3	1	$P$	$-2P$
4	-1	$2P$	$3P$
5	1	$-P$	$-3,5P$

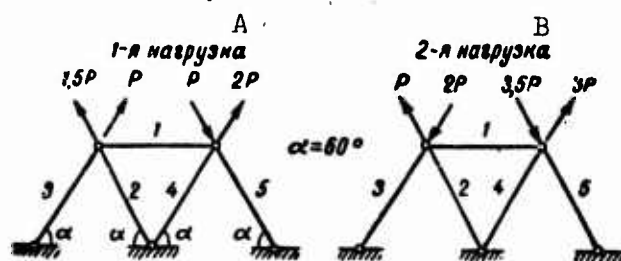


Fig. 1. A) 1st load; B) 2nd load.

A) Bar number.

For a once statically indeterminate truss the regions of feasible forces are segments on the force axis (one-dimensional space), separated on it by the points

$$N_{ik} = a_{i1} X_{1k} + b_{ik} = 0.$$

The regions of feasible values of the redundant unknown for the first load are shown in Fig. 2a. Near each boundary separating one region from another or from unfeasible values  $X_{11}$  is a circle containing the number of the bar which gives this boundary. The force in redundant bar  $X_{11}$  on this drawing is interpreted by the coordinate describing the points, while the forces in the remaining bars are proportional to the distance from it to the corresponding boundaries.

All that was said about Fig. 2a pertains also to Fig. 2b, which gives the regions of feasible forces due to the second load.<sup>1</sup>

We take on the plane mutually perpendicular axes  $OX_{11}$  and  $OX_{12}$

(Fig. 3) and draw the following: on axis  $OX_{11}$  the boundaries of regions of feasible forces due to the first load, on axis  $OX_{12}$  — these boundaries due to the second load. The straight lines drawn through the extreme boundaries, i.e., points  $E$  and  $U$  on axis  $OX_{11}$  and points  $E$  and  $K$  on axis  $OX_{12}$  parallel to the coordinate axes separate rectangle  $EKVU$ , i.e., a closely enveloping rectangle.

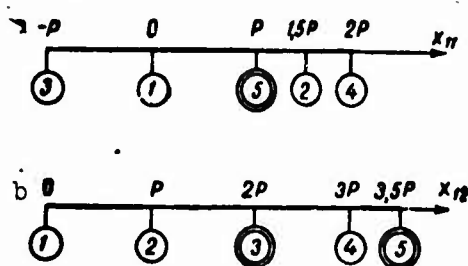


Fig. 2

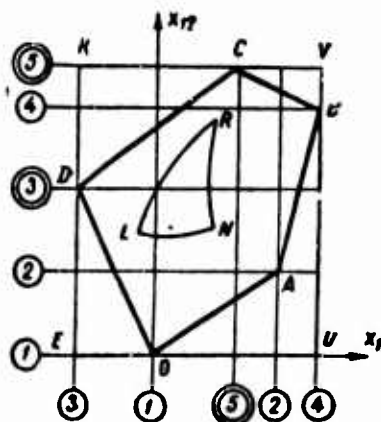


Fig. 3

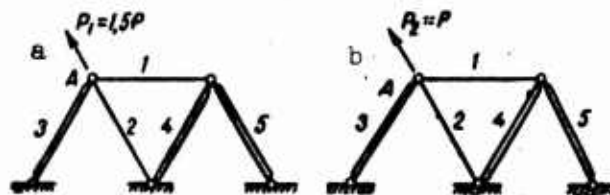


Fig. 4

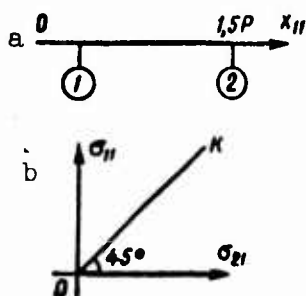


Fig. 5

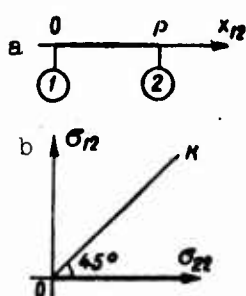


Fig. 6

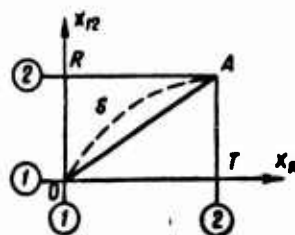


Fig. 7

It is natural to regard the region of feasible values of  $X_{11}$  and  $X_{12}$  for a truss subjected to two nonsimultaneous loads that part of the  $X_{11}, X_{12}$  plane which contains points with coordinates  $X_{11}$  and  $X_{12}$  for all the possible cross-sectional areas of its bars up to infinitely large, but not more than in  $m-(n+1)$  bars simultaneously, since in the opposite case the truss will become an infinite rigid body.

This region cannot go past the limits of the enveloping rectangle, since this would mean that it is possible to design a

truss for at least one load outside the ensemble of regions of feasible forces due to this load. At the same time the contour should be in contact with all the sides of this rectangle, since in the opposite case it would have been impossible to design a truss for at least one of the loads in each point of existence of its forces due to this load. This is illustrated graphically by Fig. 3. The region of feasible  $X_{11}, X_{12}$  cannot lie within the limits of the curvilinear contour  $LRN$ . Candidates for this region may be: polygon  $ODCBA$ , or that part of rectangle  $EKVU$  which is situated outside of this polygon, or a part of the plane situated between the broken contour  $ODCBA$  and the curvilinear contour  $LRN$ .

We have thus established the necessary attribute of the region of feasible forces  $X_{11}, X_{12}$ : it must be in contact with all the sides of the enveloping rectangle.

It is expedient to start our further discussion of the nature of the sought region and of the method for finding it from the simplest, once statically indeterminate truss, consisting of two elastic bars only.

Let this be the truss of Fig. 4. It was obtained from the truss of Fig. 1 on the assumption that all the bars of the latter, with exception of the first and second, are infinitely rigid,<sup>2</sup> while the forces directed along these infinitely rigid bars have been dropped, since they do not affect the forces in the elastic bars.

The region of feasible forces due to the first load is shown in Fig. 5a, while this region due to the second force is shown in Fig. 6a. The truss has one primary bar (the infinitely rigid bars are not counted), hence the plane depicts also regions of feasible forces [3]: for the first load the feasible stresses are given by the positive half axis  $O\sigma_{21}$  (Fig. 5b), while for the second these are given by the positive half axis  $O\sigma_{22}$  (Fig. 6b).

In Fig. 7 the coordinate axes  $OX_{11}, OX_{12}$  with the regions of feasible forces due to each of the loads depicted on them are located mutually perpendicular. This figure also shows enveloping rectangular  $ORAT$ .

In a two-bar truss the forces from any load are functions of one parameter only, i.e., of the area ratio  $\frac{F_2}{F_1}$ . Hence  $X_{11} = f_1\left(\frac{F_2}{F_1}\right)$  and  $X_{12} = f_2\left(\frac{F_2}{F_1}\right)$ . These two relationships yield a line on the  $X_{11}, X_{12}$  plane. A segment of this line, according to the necessary attribute, must be in contact with the enveloping rectangle. Let us find its ends.

Holding  $F_2$  constant and assuming  $F_1 \rightarrow 0$ , we will get a statically indeterminate truss in which  $N_{11} = 0$  and  $N_{12} = 0$  (bar 1 will disappear). In Fig. 7 this truss is depicted by point  $O$ . One end has been found. Assuming then that  $F_1$  is constant and  $F_2 \rightarrow 0$ , we get a statically indeterminate truss in which  $N_{21} = 0$  and  $N_{22} = 0$ . This is point  $A$ , which is also the second end.

What does this sought linear segment represent?

If we assume that it is curved, let this be, for example, segment  $OSA$ ; then we will contradict the equally justified mutual location of the coordinate axes and the segment, which follows from the equivalence of both loads. In fact, why should the sought linear segment be directed by its bulge toward the positive half axis  $OX_{11}$  and not toward the positive half axis  $OX_{12}$ ? There exists only one linear segment which satisfies the above equivalence, and this is the rectilinear segment  $OA$ .

The region of feasible forces for a truss from two elastic bars is thus a rectilinear segment, a diagonal of the enclosing rectangle.

For the particular case under consideration of a truss from two bars (Fig. 4), when the forces in its bars are produced only by a force applied in node  $A$ , it is possible to give a simpler proof of the above result.

In fact, no matter in which manner the force in node  $A$  is placed, it can be broken up into the directions of bars 2 and 3. The component along bar 3 can be disregarded, since this bar is infinitely rigid. Forces in bars 1 and 2 may be produced only by the load component along bar 2. Hence Figs. 4a and 4b depict the most general case of two nonsimultaneous loads on the truss under consideration. But it may be claimed with equal justification that these are not two loads, but one, except that the load-specifying parameter has changed. Load  $P_2$  is load  $P_1$  reduced by a factor of one and one half.

Therefore

$$\frac{X_{12}}{X_{11}} = \frac{P_2}{P_1}$$

for all the possible values of ratio  $\frac{F_2}{F_1}$ . On the  $X_{11}, X_{12}$  plane this is an equation of a straight line.

The latter proof is regarded as particular, since not in all trusses from two elastic bars two different loads are forces applied in the same node.

Then for any internal point of segment  $OA$  forces  $X_{11} \neq 0$  and  $X_{12} \neq 0$ , and therefore when specifying  $\sigma_{21}$  on selecting a truss it is impossible to specify  $\sigma_{21} = 0$ , since then we will get  $\sigma_{11} = 0$  (see Fig. 5b) and the truss will have  $F_1 = \infty$  and  $F_2 = \infty$ . Such a truss (infinite rigid body) is excluded from the set of trusses under consideration. When  $\sigma_{21} > 0$  it is found that  $\sigma_{11} > 0$ , all the internal points of segment  $OA$  thus yield ordinary trusses with finite areas  $F_1$  and  $F_2$ .

As to the trusses given by points  $O$  and  $A$ , their features will depend on the method of specifying the describing points of forces and stresses in its regions. If we specify the forces due, for example, to the first load, by point  $T$ , and stresses  $\sigma_{21} > 0$

and we take  $\sigma_{11} = \sigma_{21}$  (this follows from Fig. 5b), then we get

$F_1 = \frac{X_{11}}{\sigma_{11}} = \frac{1.5P}{\sigma_{11}} \neq 0$ , while  $F_2 = \frac{N_{21}}{\sigma_{21}} = \frac{0}{\sigma_{21}} = 0$ . This truss was mentioned previously, it is regarded as statically indeterminate [1], since the stresses in it are related by the strain compatibility condition. However, the forces in it are the same as those in the statically determinate truss which is obtainable from the given statically indeterminate truss (Fig. 4) by removing bar 2. If we do not require that  $\sigma_{11}$  satisfy the strain compatibility equations, then point A will yield an authentic statically determinate truss.

If, as previously, the forces are specified by point T, we assume  $\sigma_{21} = 0$  and get  $\sigma_{11} = 0$ ; then  $F_1 = \frac{X_{11}}{\sigma_{11}} = \frac{1.5P}{0} = \infty$ , while  $F_2 = \frac{N_{21}}{\sigma_{21}} = \frac{0}{0}$ . Area  $F_2$ , being indeterminate, can be specified at will. But we have agreed not to consider such trusses. Hence it is better to assume that the truss does not have bar 2, and then the area of bar 1 may, if desired, be also regarded as infinitely large. Although this will make the truss an infinitely rigid body, no unclarities will exist with respect to the force distribution in it.

Since in the truss described by point 0,  $N_{11} = 0$  and  $N_{12} = 0$ , while in that described by point A,  $N_{21} = 0$  and  $N_{22} = 0$ , then it is convenient to denote on the  $X_{11}, X_{12}$  plane point 0 by the number 1 and point A by the number 2. This notation will remind us that one boundary of the region of feasible forces is obtained as a result of drawing lines through point 1 on both axes, while the other is obtained by drawing lines through point 2. The first of them will be determined by the values of  $X_{11}$  and  $X_{12}$  in a truss with bar 1 missing, while the other will be obtained from these values in a truss without bar 2. The region of feasible forces, segment OA, is consequently obtained from equations of statics.

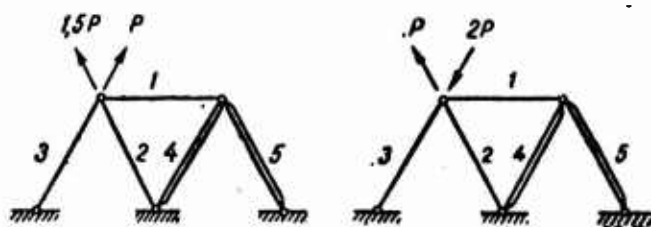


Fig. 8

We now complicate the truss (Fig. 8) as follows: we assume  $m = 3$ . The regions of feasible forces due to each of the loads on the  $OX_{11}$  and  $OX_{12}$  axes and the enveloping rectangle EDTL for it are shown in Fig. 9. The regions of feasible stresses for the first load are shown in Fig. 10, while those for the second load are in Fig. 11.

If in the truss (Fig. 8), starting with some finite areas  $F_1, F_2, F_3$ , we vary only  $F_1 \rightarrow \infty$ , then we will get a truss with two elastic bars 2 and 3. According to previous results, its region of feasible values of  $X_{11}$  and  $X_{12}$  in Fig. 9 is given by segment 2-3.

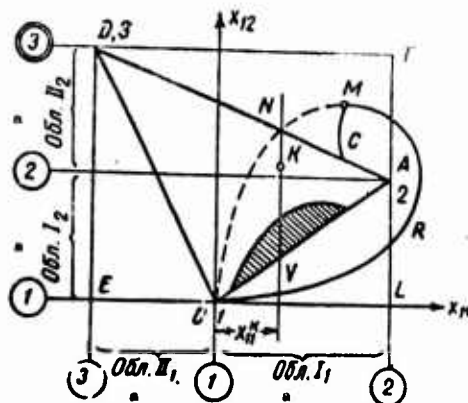


Fig. 9. a) Region.

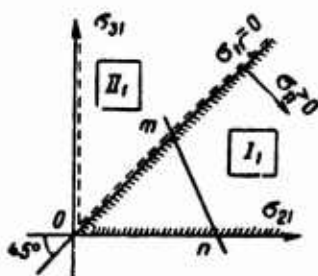


Fig. 10

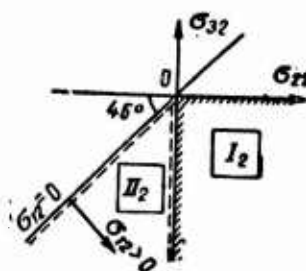


Fig. 11

If we assume that only  $F_2 \rightarrow \infty$ , then it is obviously given by segment 1-3; if  $F_3 \rightarrow \infty$ , then it is given by segment 1-2.

We have found the boundaries of the sought region. They form a closed figure – the triangle 123 or in alphabetical notation – triangle  $OAD$ . What is the region proper?

We now prove that the sought region cannot go past the limits of triangle  $OAD$ . This will be done by assuming the converse. Let us assume that for some finite values of areas  $F_1 = F_1^M$ ,  $F_2 = F_2^M$  and  $F_3 = F_3^M$  the describing point  $M$  lies, for example, inside triangle  $ADT$  (Fig. 9). Then holding  $F_2$  and  $F_3$  constant and constantly varying the area of the first bar within the limits  $F_1^M < F_1 < \infty$ , we will get a truss with an infinitely rigid bar 1. This continuous variation of area  $F_1$  has corresponding to it the motion of the describing point  $(X_{11}, X_{12})$  along some continuous curve from point  $M$  to some point  $C$  onto line  $AD$ , but not onto its ends.

The process of area variation considered above does not contradict the location of points  $M$  outside of triangle  $OAD$ . But also a second process is possible whereby  $F_2 = F_2^M$ ,  $F_3 = F_3^M$ , while  $0 \leq F_1 < F_1^M$ . The statically indeterminate truss obtained as a result of this process does not differ with respect to the force distribution from that statically determinate truss which is produced from the specified truss by removing bar 1. The forces in this truss are described by point  $O$  (Fig. 9). The describing point should therefore, moving along a continuous line, move from point  $M$  into point  $O$ . Such a transition is possible either along a line such as  $MRO$ ,

not intersecting segment  $AD$ , or along a line such as  $MNO$  (dashed line), intersecting segment  $AD$ . The first path is forbidden, since it removes us past the limits of the enveloping triangle. The second path requires moving onto line  $AD$ , and this is only possible as a result of  $F_3 \rightarrow \infty$ , i.e., as a result of a process which is directly opposite to the process  $F_3 \rightarrow 0$ . We have arrived at a contradiction and, consequently, point  $M$  cannot lie inside triangle  $ADT$ . The possibility that point  $M$  is located inside triangles  $OAL$  and  $ODE$  is disproved similarly.

Thus, the region of feasible forces lies inside triangle  $OAD$ . We now show that it completely covers this triangle.

Let us assume that triangle  $OAD$  has a void (crosshatched).

Selecting a truss for the first load by the method of specified stresses, we must specify three parameters:  $X_{11}$ ,  $\sigma_{21}$  and  $\sigma_{31}$ . We take in Fig. 9 point  $K$  so that the straight line  $KV$ , which is parallel to axis  $OX_{12}$ , would intersect the assumed void. We assume  $X_{11}^* = X_{11}^k$ . The value of  $X_{11}^k$  belongs to force region  $I_1$ . The values of  $\sigma_{21}$  and  $\sigma_{31}$  must therefore be selected from stress region  $I_1$  (Fig. 10). We draw in it some line intersecting its boundaries; let it be straight line  $mn$ . Specifying as the given stresses  $\sigma_{21}$  and  $\sigma_{31}$  the coordinates of any point of segment  $mn$ , we will get a set of trusses in which  $X_{11} = X_{11}^k$ . Points  $(X_{11}^k, X_{12})$  on the  $X_{11}, X_{12}$  plane corresponding to these trusses will lie on segment  $NV$  without moving past its limits, since there are no feasible  $X_{11}, X_{12}$  outside of triangle  $OAD$ . The stress point  $m$  has  $\sigma_{11} = 0, \sigma_{21} \neq 0, \sigma_{31} \neq 0$ . Since  $N_{11}^k \neq 0, N_{21}^k \neq 0$  and  $N_{31}^k \neq 0$ , we get a truss in which  $F_1 = \infty$ , while  $F_2$  and  $F_3$  are finite. At segment  $NV$  this truss will be described by point  $N$ , as a truss with infinitely rigid bar 1 and elastic bars 2 and 3. The stress point  $n$  has corresponding to it the truss described by point  $V$ .

The continuous motion of the stress point along segment  $mn$  from point  $m$  to point  $n$  has corresponding to it a continuous motion of the force point from point  $N$  to point  $V$  along segment  $NV$ . Consequently, all the points on segment  $NV$  yield feasible forces, which means that the assumption of a void is incorrect.

Thus, for a truss with  $m = 3$ , the regions of feasible forces  $X_{11}, X_{12}$  is triangle 123.

Finding of its vertices is, obviously, a statically determinate problem.

2. We now consider a truss with any number of elastic bars. Let this be a truss with  $m = 5$  (Fig. 1).

We draw on axis  $OX_{11}$  the regions of feasible forces due to the first load and the regions of feasible forces due to the second load — on axis  $OX_{12}$  (Fig. 12). Coordinate lines are drawn through the extreme points of both axes. These limiting lines separate the enveloping rectangle, while the intermediate lines break it up into individual rectangles. The points formed by intersection of lines with subscripts  $i$  and  $j$  will be denoted by  $(i, j)$ . Among them will also be point  $(i, i)$ , which, for brevity, we have

already agreed to denote by a single  $i$ . We have five such points: 1, 2, 3, 4, 5. We connect pairs of these points by lines. A part of them will isolate a close convex polygon. Obviously it will be in contact with all the sides of the enveloping polygon.

We now show that the above polygon is a region of feasible forces due to two loads.

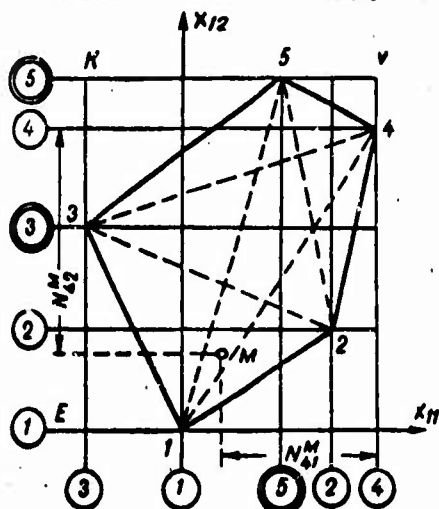


Fig. 12

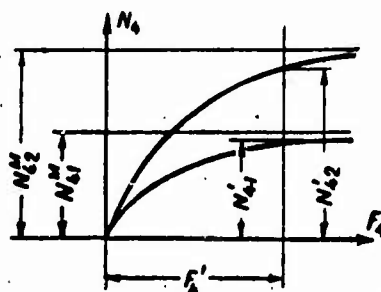


Fig. 13

First of all we note that it satisfies the necessary attribute. We then imagine that in the truss of Fig. 1 first bar 5 has hardened (which is the same as  $F_5 = \infty$ ), and then bar 4. We get a truss from three elastic bars 1, 2, 3. The region of feasible forces for it is triangle 123. On Fig. 12 it will definitely be a part of the previously constructed polygon, since the latter was constructed precisely as a collection of such overlapping triangles.

We take inside triangle 123 point  $M$  (Fig. 12). It will have corresponding to it some finite areas  $F_1^M, F_2^M, F_3^M$  and, if desired, reference may be had to  $F_4 = \infty$  and  $F_5 = \infty$ .

The forces in the bars of this truss due to both loads are proportional to the distances from point  $M$  to the corresponding boundaries (circles). In particular in infinitely rigid bar 4 we denote them by  $N_{41}^M, N_{42}^M$  (see Fig. 12).

We now imagine that bar 4 ceased to be perfectly rigid, and we consider now the set of trusses with  $F_1 = F_1^M, F_2 = F_2^M, F_3 = F_3^M$ , while the area of the fourth bar varies in the entire range  $0 < F_4 \leq \infty$ .

It is known that then  $N_4$  will behave according to the graph of Fig. 13. It follows from the nature of this graph that it is always possible to find a value of  $F_4$ , which we shall call  $F_4'$  (see Fig. 13) such, for which forces  $N_{41}'$  and  $N_{42}'$  in a truss with bar areas  $F_1 = F_1^M, F_2 = F_2^M, F_3 = F_3^M$  and  $F_4 = F_4'$  will differ as little as desired from  $N_{41}^M$  and  $N_{42}^M$ . But, in a once statically indeterminate truss a

small difference between forces due to a certain load in one bar results in a small difference in the remaining bars, in particular in the first, the forces in which determine point  $M$  on the  $X_{11}, X_{12}$  plane. We have shown by virtue of this that in an as small as desired vicinity of point  $M$  it is possible to design a truss with four elastic bars.

Consequently, the region of feasible forces in a truss with three elastic bars and with the fourth infinitely rigid is at the same time a region of feasible forces also for a truss with four elastic bars. But the same considerations can be sequentially extended also to all the remaining bars which were previously imagined to be perfectly rigid.

The entire polygon 12345 of Fig. 12 can be broken up by diagonals into triangles. Each of them, being a region of feasible forces for a truss from the corresponding three elastic bars, is simultaneously also a region of feasible forces for the specified truss. Consequently, polygon 12345 is the region of feasible forces for a once statically indeterminate truss, operating under two loads. The procedure for constructing it has been described above. It is constructed by means of equations of statics.

Each side of polygon 12345 lies on a separate line. Cases are obviously possible when several sides lie on the same straight line.

## §2. A Once Statically Indeterminate Truss Subjected to an Arbitrary Number of Loads

1. We now formulate the necessary attribute of a region of feasible forces for any number  $v$  of loads.

For  $v = 3$  in the  $X_{11}, X_{12}, X_{13}$  space we may have reference to an enveloping parallelepiped, analogous to the manner in which for  $v = 2$  we referred to an enveloping rectangle. The necessary attribute consists in the fact that the region of feasible forces should be such an ensemble of points of space  $X_{11}, X_{12}, X_{13}$  which, firstly, does not go past the limits of the enveloping parallelepiped and, secondly, is in contact with all of its boundaries. This follows from the concept of regions of feasible forces for a single load. In the case of any number of loads one must add to this formulation that reference is had to corresponding patterns in a  $v$ -dimensional space.

2. If we examine all the considerations which enabled us to find the region of feasible forces for  $v = 2$ , i.e., for a two-dimensional space, then it will become clear that they can be extended to the case of any number of loads  $v$ . It only remains to formulate the corresponding results for the case of a multidimensional space.

*The number of elastic bars in the truss is  $m = 2$ .*

For any  $v$  the region of feasible forces is a rectilinear segment.

*In the truss  $m = 3$ .*

When  $v \geq 2$  the region of feasible forces is a triangle. Its boundaries are the vertices and the sides.

*In the truss  $m = 4$ .*

When  $v \geq 3$  the region of feasible forces is a tetrahedron. Its boundaries are the vertices, sides and triangular faces.

*In the truss  $m = 5$ .*

When  $v \geq 4$  the region of feasible forces is a bounded part of a four-dimensional space. Its boundaries are vertices (zero-dimensional boundaries), sides (one-dimensional boundaries), triangular faces (two-dimensional boundaries) and tetrahedra (three-dimensional boundaries).

*The truss has an arbitrary  $m$ .*

When  $v \geq m-1$  the region of feasible forces is a bounded part of an  $(m-1)$ -dimensional space. It has boundaries of all dimensions from zero to  $m-2$  inclusively.

The boundaries with the highest dimensionality will be called principal boundaries.

The most complete definition of a region of feasible forces, which is a convex body in a multidimensional space, consists in knowing its boundaries up to the principal boundaries. However, in applications it is possible to restrict oneself to something less than this, i.e., knowing the vertices and principal boundaries, since even with this information it is easy to take any point in the convex polyhedron, which is precisely what is needed in selecting a truss.

When  $v \geq m-1$ , finding of regions of feasible forces reduces to finding  $m$  of its vertices. This is so since in the  $v=m-1$  dimensional space the boundaries of all the dimensions drawn through  $m$  vertices separate an elementary convex polyhedron. Each of the combinations  $C_m^2$  yields one-dimensional boundaries of this polyhedron (sides), each of combinations  $C_m^3$  yields two-dimensional boundaries (triangular sides), each of the combinations  $C_m^4$  yields three-dimensional boundaries (tetrahedra), etc., up to the principal boundaries of the  $m-2$  dimension. The number of the latter is obviously  $C_m^{m-1} = m$ . All these boundaries, which contain the most complete information about the polyhedron, are obtained without calculations (this remark does not pertain to the vertices). It is hence necessary to describe a method for finding the boundaries of the region only for  $v < m-1$ .

We first find its vertices. For this we drop in the specified truss the  $i$ th elastic bar and in the statically determinate truss thus obtained we find  $X_{11}^{(i)}, X_{12}^{(i)}, \dots, X_{1n}^{(i)}$ . This will be the coordinates of the  $i$ th vertex of the sought polyhedron. Obviously, the number of

vertices is equal to the number of all the elastic bars, i.e.,  $m$ .

Through each  $v$  vertices we draw a  $v - 1$ -dimensional plane. Its equation can be written in the form

$$D = \begin{vmatrix} X_{11} & X_{12} & X_{13} & \dots & X_{1v} & 1 \\ X_{11}^{(1)} & X_{12}^{(1)} & X_{13}^{(1)} & \dots & X_{1v}^{(1)} & 1 \\ X_{11}^{(2)} & X_{12}^{(2)} & X_{13}^{(2)} & \dots & X_{1v}^{(2)} & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ X_{11}^{(v)} & X_{12}^{(v)} & X_{13}^{(v)} & \dots & X_{1v}^{(v)} & 1 \end{vmatrix} = 0.$$

We will have  $C_m^v$  such planes.

Let us determine on which of these do the principal boundaries of the region lie. If a principal boundary lies on a plane, then all the remaining vertices of the region (amounting to  $m-v$ ), by virtue of its convexity, will lie to one side of this plane. This is expressed analytically by the fact that determinant  $D$  for all the vertices has the same sign.

After finding all the principal boundaries of the region of feasible forces it is easy to specify any point from this region. We take any two vertices  $i$  and  $j$  which do not lie on the same principal boundary. Then any internal point will be found from the formula

$$X_{1k} = X_{1k}^{(i)} + \alpha (X_{1k}^{(j)} - X_{1k}^{(i)}); \quad 0 < \alpha < 1, \quad k = 1, 2, 3, \dots, v. \quad (1)$$

A point in a polyhedron may also be specified as follows. The "center of gravity" of the vertices of a convex polyhedron is found from the formula

$$X_{1k} = \frac{\sum_{i=1}^{i=m} X_{1k}^{(i)}}{m}, \quad k = 1, 2, 3, \dots, v.$$

It is an internal point of the polyhedron. On the segment connecting this center with any vertex of the polyhedron it is possible to select a point using Expression (1).

3. The first solution of the problem of the region of feasible forces for several loads was published by K.M. Khuberyan [7].

Due to the nature of the region of feasible forces he has arrived at the following two conclusions, first of which is quoted directly from [7]:

"...when designing a truss for two loads the region of existence of redundant unknowns  $X^I$  and  $X^{II}$  is a part of a rectangle, cut off from it by some straight line; analogously to it, when designing a truss for three loads the region of existence of the redundant unknowns  $X^I$ ,  $X^{II}$  and  $X^{III}$  is a part of a parallelepiped, which is cut off from it by some plane; finally, in the general case of designing a truss for  $\mu$  loads, the region of existence of the redundant un-

knowns  $X^I, X^{II}, \dots, X^{III}$  is a part of a  $\mu$ -dimensional parallelotope, cut off from it by some  $\mu - 1$ -dimensional hyperplane."

The rectangle, parallelepiped and parallelotope mentioned above are, according to this article's terminology, the enveloping rectangle, enveloping parallelepiped and enveloping multidimensional "parallelepiped."

In addition,  $\mu$  is our  $v$ ,  $X^I$  is our  $X_{11}$ ,  $X^{II}$  is our  $X_{12}$ , etc.

K.M. Khuberyan's second conclusion consists in the fact that the above straight line, plane and hyperplane cannot be found from the equilibrium equation. In other words, the finding of a region of feasible forces for  $v \geq 2$  is a statically indeterminate problem.

However, in view of this author's study it is clear that the region of feasible forces is separated from the enveloping rectangle in the general case not by one straight line, but by a broken line (see, for example, Fig. 12), and secondly, the region of feasible forces can always be found from equations of statics.

### §3. Trusses with any Number of Redundant Couplings

1. In an  $n$ th-order statically indeterminate truss the forces due to one load are determined by  $n$  redundant unknowns. Forces due to  $v$  successive loads will be determined by  $nv$  unknowns. Hence the region of feasible forces is a limited set of points in an  $nv$ -dimensional space.

For  $v = 1$  and any  $n$  the problem of regions of feasible forces is solved in [3]. In particular, for  $n \leq 2$  all the regions of feasible forces are described on a plane and it is hence easy to find them for any number of elastic bars in a truss.

Below we shall illustrate certain computations by a twice statically indeterminate truss (Fig. 14). Its first load is shown in Fig. 14a and the second in Fig. 14b. Table 2 shows forces  $a_{11}, a_{12}, b_{11}, b_{12}$ . It was assumed that the redundant bars are 1 and 2. The regions of existence of forces due to the first load are depicted in Fig. 15; those for forces due to the second load are given in Fig. 16.

2. In order that the truss for  $n = 2$  would not act as a perfect rigid body, it should have at least three elastic bars. The needed truss and its loads can be obtained from Fig. 14 by setting  $F_4 = \infty$ .

Thus, let us take a truss with  $m = 3$ .

The number of loads is  $v = 2$ . The space in which the region of feasible forces is sought is  $nv=4$  dimensional.

The forces in a truss with three elastic bars depend on two parameters. Consequently, the ensemble of the feasible  $X_{11}, X_{21}$  and  $X_{12}, X_{22}$  is given by a two-dimensional set of points, i.e., surface, in the four-dimensional space.

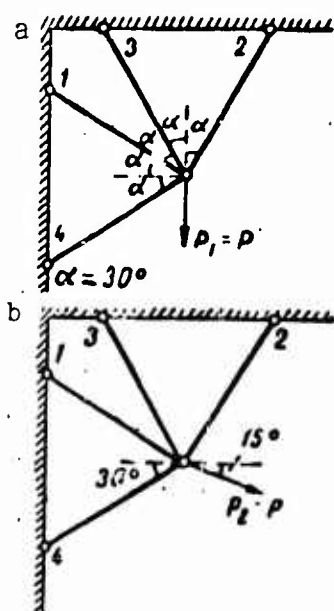


Fig. 14

TABLE 2

№ стержней A	$a_{11}$	$a_{12}$	$b_{11}$	$b_{12}$
1	1	0	0	0
2	0	1	0	0
3	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}P$	$\frac{\sqrt{2}}{2}P$
4	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}P$	$\frac{\sqrt{2}}{2}P$

A) Bar numbers.

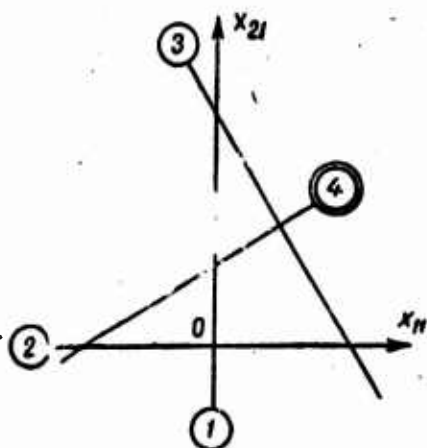


Fig. 15

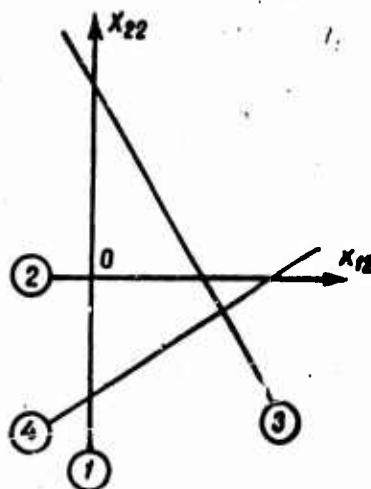


Fig. 16

Among statically indeterminate trusses with three elastic bars we think of trusses with disappearing areas in each bar in turn.

Let, for example,  $F_3=0$  (then  $F_4=\infty$ ). This is substantially a truss with one redundant and two elastic bars. The ensemble of forces feasible in it, as established in §1, is a rectangular segment. One of its ends corresponds to a truss with  $F_1=0$ , while the other end corresponds to a truss with  $F_2=0$ . But since then  $F_3=0$ , then in a four-dimensional space one end of the segment corresponds to a truss with missing bars 1 and 3, while the other corresponds to a truss with bars 2 and 3 missing. The points proper are denoted by (13) and (23), while the segment is designated as (13)-(23).

Let then only  $F_2=0$ . The feasible forces cover the segment (12)-(23). Finally, when  $F_1=0$ , they cover the segment (12)-(13).

The ends of the segments show that the segments form a triangle (linear). The zero- and one-dimensional boundaries of the sought surface have been found. Symmetry considerations force the conclusion that this is a plane.

Thus, for  $v = 2$  (we recall that  $n=2$  and  $m=3$ ) the region of feasible forces is a triangle. Its vertices correspond to all the statically determinate trusses which can be obtained from a given truss having three elastic bars by removing two of them.

For the specific truss under study (Fig. 14,  $F_4=\infty$ ) the projection of this triangle onto the  $X_{11}, X_{21}$  plane is obtained from Fig. 15, while its projection onto the  $X_{12}, X_{22}$  plane is obtained from Fig. 16 by removing line 4 from them.

When  $v > 2$  the region of feasible forces is the triangle (12) (13) (23) in the  $2v$ -dimensional space.

The truss has  $m = 4$ .

The forces in it depend on three parameters; therefore the region of feasible forces will be bound by a three dimensional set.

As before the number of loads is  $v = 2$ . Let us clarify the two-dimensional boundaries of this set.

We set  $F_1=\infty$ . The truss will consist of elastic bars 2, 3 and 4. The feasible forces are given by points of the triangle (23) (24) (34). It is the first sought two-dimensional boundary.

$F_2=\infty$ , triangle (13) (14) (34) is the second boundary;

$F_3=\infty$ , triangle (12) (14) (24) is the third boundary;

$F_4=\infty$ , triangle (12) (13) (23) is the fourth boundary.

These four triangles may isolate in a four-dimensional space a three-dimensional region, but the triangles found above will not do this, since they have coinciding vertices, but no coinciding sides. This points to the fact that not all the two-dimensional boundaries have been found.

In a truss with  $m = 4$  we may set in sequence the area of each bar equal to zero.

Let  $F_1=0$ . Then the truss will consist of three elastic bars 2, 3 and 4, from which one is redundant. The feasible forces for it lie on a triangle. Its vertices correspond to trusses in which at first only  $F_2=0$ , then only  $F_3=0$  and, finally, only  $F_4=0$ . But since here all the time also  $F_1=0$ , then in the four-dimensional space the triangle has vertices (12), (13) and (14). This will be still another boundary. We may therefore claim that:

$F_1=0$ , triangle (12) (13) (14) is the fifth boundary;

$F_2=0$ , triangle (21) (23) (24) is the sixth boundary;

$F_3=0$ , triangle (31) (32) (34) is the seventh boundary;

$F_4=0$ , triangle (41) (42) (43) is the eighth boundary.

These eight triangles isolate some closed three-dimensional polyhedron in the four-dimensional space.

Its projection on the  $X_{11}, X_{21}$  coordinate plane is described in Fig. 15, while the projection on the  $X_{12}, X_{22}$  plane is given in Fig. 16.

These projections are not convex polygons. Consequently, the polyhedron thus found is not convex. This is a new property of the region of feasible forces, which it did not have for  $n = 1$ .

If for  $n=2$  and  $v=2$  and the number of bars in the truss is  $m = 5$ , then the region of feasible forces will already be four-dimensional. In general, in order that in a twice statically indeterminate truss for any  $v$  the region of feasible forces have the dimensions of the space in which it is situated, it is necessary to satisfy the condition  $m \geq 2v+1$ .

3. As we have seen above, from the fact that the problem of the region of feasible forces for any  $v$  for  $n=1$  was solved by means of equations of statics, it followed that it is solvable by the same approach also for  $n=2$ . The same can be discovered for  $n$  and  $n+1$ . In other words, the finding of the region of feasible forces for any  $n$  and  $v$  is a statically determinate problem.

The vertices of the region are easily found. The fact that for  $n \geq 2$  this region is not a convex body appreciably complicates the finding of its remaining boundaries. The author expects to return to this problem.

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#### Footnotes

- 169 <sup>1</sup>The rule of signs for the forces of Fig. 2 and other details are given in [3].
- 171 <sup>2</sup>The infinitely rigid bars are depicted by double lines.

## A STUDY OF THE FUNCTIONING OF THE TRANSVERSE STRUCTURE OF A LARGE-PANEL BUILDING

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The modern large-panel house is a complex statically indeterminate system, consisting of deformable elements connected by compliant couplings. The need for taking into account the compliance of the couplings, as well as the effect on the compression and shear of the wall panels of production-process cracks and poor strength of concrete in tension was noted in a number of studies [1, 2].

One of the most interesting attempts to bring the computational scheme closer to actual conditions of the static functioning of a large-panel house was made by B.A. Kositsyn [3]. However, in determining the rigidity in shear of wall structures, he has not taken into account the compliance of the couplings; in determining the rigidity in shear the panels were basically assumed to be perfectly rigid, while the compliance of the couplings was determined on the basis of the cross sections of connecting accessories in the panels' connecting strips. One must also disagree with the assumption made in this study that the rigidity of the tension zone of the panels is determined solely by the cross section of the reinforcements, i.e., that the concrete ceases entirely to function.

The method of design of the transverse structure of the series K-7 building [5], recommended in instructional literature [4], is based on the assumption that the thrust produced by a load applied after welding the seams in the load-bearing cross sections is constant along the height. The compliance of couplings and of the panels is here taken into account by a single empirical coefficient.

In seeking correct ways of study and design of large-panel buildings, it is first of all necessary to forgo the estimating of the stiffness of the building's elements on the basis of the elastic properties of the starting materials, which are found by testing small standard specimens (cubes, prisms, small beams, etc.), since the functioning of panels depends on a large number of factors which cannot be taken into account in these tests.

It is possible to come substantially close to reality when

setting up a computational system by making use of the actual deformational properties of the panels, which are fully characterized by the matrix of the translations of points on its contour.

Below is considered the design of panel structures, based on the use of matrices of this kind and making it possible to take into account the compliance of couplings without substantial complications. As an example, we shall consider the transverse structure of one of the most massive buildings - the K-7.

### §1. Design of the Transverse Structure of the K-7 Building

This building has load-bearing transverse structures with a spacing of 3.2 m and outside curtain walls. The transverse structure is formed by a system of thin-walled panels, which have thickenings along the contour. The floor structures are supported by the lower flange. The design scheme of the structure for the case of a vertical axis of symmetry is shown in Fig. 1a.

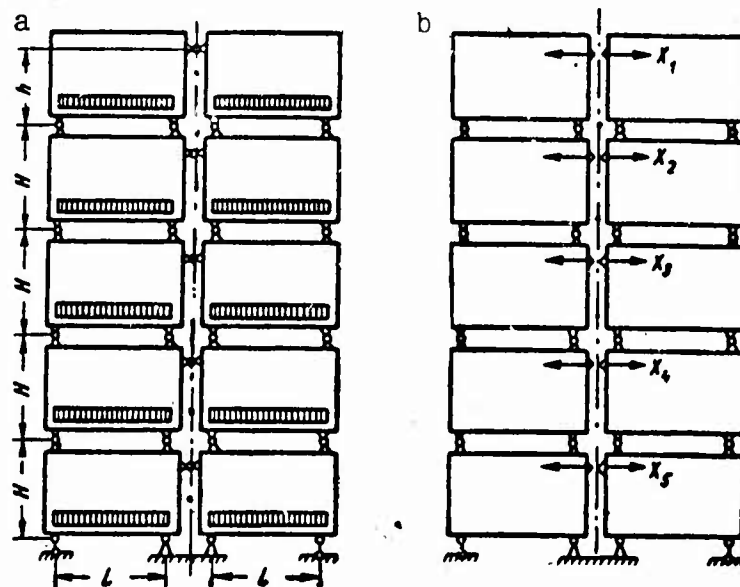


Fig. 1

For simplicity we assume that: a) all the couplings are linearly deformable, b) the load and settling of the supports are uniformly distributed in the longitudinal direction.

We restrict ourselves to symmetrical actions. As the principal system we take the statically indeterminate system shown in Fig. 1b.

**Rigidity of couplings.** Panels of the same story are interconnected by "lateral couplings" (Fig. 2a), while panels of different stories are connected by "end" couplings shown in a section through panels along the median plane in Fig. 2b.

The compliance of couplings consists of the deformation of the fastening accessories and the compliance of their fastening

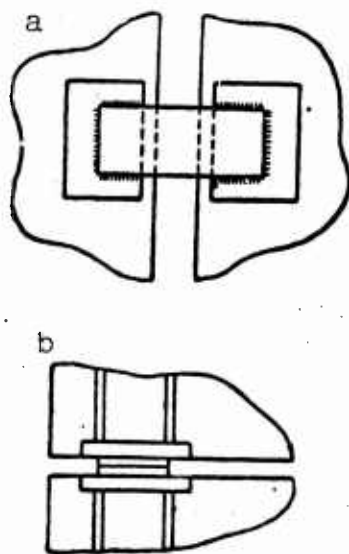


Fig. 2

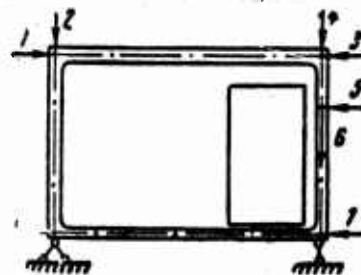


Fig. 3

in the concrete. For symmetrical forces, it is characterized by three specified parameters:  $\Delta_g$ , which is the horizontal elongation of the end coupling due to a unit horizontal force applied to it,  $\Delta_v$ , which is the vertical elongation of the end coupling due to a unit vertical load applied to it, and  $\Delta_b$ , the horizontal elongation of the lateral coupling due to the unit horizontal force applied to it.

**Stiffness of the panels.** The stiffness of the panels is fully defined by the displacements of points on the contour in which the couplings are situated, under the action of forces applied in these points, as well as due to the load distributed along the lower flange. Figure 3 shows the positive directions of the corresponding forces and translations.

Henceforth we shall assume that all the values  $\Delta_{ik}$  of translations in the  $i$ th point of the contour in the  $i$ th direction, due to a unit force applied in the  $k$ th point of the contour in the  $k$ th direction are known.

**Distribution of horizontal forces among the couplings in the principal system.** To determine the coefficients of canonical equations it is every time necessary to design the statically indeterminate principal system for unit forces applied instead of the cut lateral couplings. Since various versions of couplings and panels are possible, this substantially complicates the calculations. It is hence expedient to introduce some additional assumptions.

We denote by  $\beta_{ik}$  the horizontal forces in the middle end couplings between the  $i$ th and  $(i + 1)$ st stories due to a unit force applied instead of the lateral couplings of the  $k$ th story. We assume that the panels situated above the panel subjected to

the load are not functioning. Then the matrix of forces  $\beta_{ik}$  will take on the form

$$\begin{vmatrix} \beta_{11} & 0 & 0 & 0 \\ \beta_{12} & \beta_{22} & 0 & 0 \\ \beta_{14} & \beta_{24} & \beta_{34} & 0 \\ \beta_{15} & \beta_{25} & \beta_{35} & \beta_{45} \end{vmatrix}$$

As a result of the fact that the system consists of identical elements, it is possible to simplify it further, which will be quite useful when working on a building with a large number of stories. It may be assumed that

$$\begin{aligned} \beta_{12} &= \beta_{22} = \beta_{34} = \beta_{45} = \beta_1; \\ \beta_{13} &= \beta_{24} = \beta_{35} = \beta_2; \\ \beta_{14} &= \beta_{25} = \beta_3; \\ \beta_{15} &= \beta_4. \end{aligned}$$

Comparison calculations which were made show that the error of this assumption is within the limits of the ordinary accuracy of calculations.

To determine the four unknown  $\beta_j$ , it is sufficient to consider the case when a single load is applied at the lateral couplings of the 5th story.

Dropping the horizontal end couplings situated on the inside, we will get the principal system for finding  $\beta_j$ , which is shown in Fig. 4.

We now find coefficients  $a_{ik}$  of this system.

For a unit load  $\beta_1=1$  forces will arise only in panels of the 4th and 5th stories and in couplings between the three upper stories. Coefficient  $a_{11}$  is found as the algebraic sum of the elongation of the bottom flange of the fifth-story panel and of the upper flange of the fourth-story panel. Upon consideration of deformation of the couplings, we get

$$a_{11} = \Delta_{11} + \Delta_{33} + 2\Delta_{13} + \Delta_{77} + 4\Delta_r.$$

Further

$$a_{21} = -\Delta_{17} - \Delta_{37}; \quad a_{31} = a_{41} = 0.$$

Having reference to the fact that for all the unit loads only two adjoining panels are functioning, we get for the remaining coefficients

$$\begin{aligned} a_{22} &= a_{33} = a_{44} = a_{11}; \\ a_{32} &= a_{43} = a_{21}; \\ a_{42} &= 0. \end{aligned}$$

Determining the mutual translations at the points of the cut end couplings due to a single force according to the diagram of Fig. 4, we find the load terms

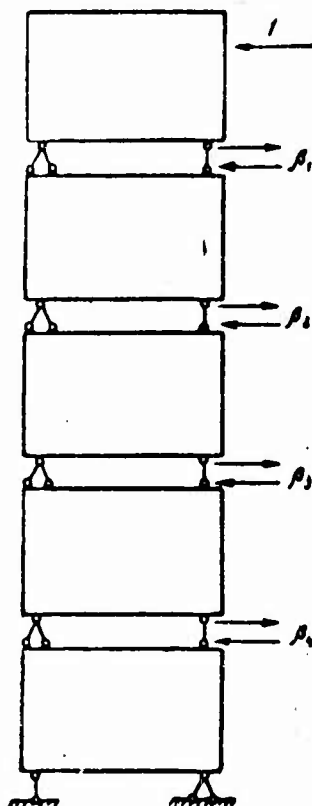


Fig. 4

$$a_{1,2} = \Delta_{11} + \Delta_{13} + \Delta_{15} + 2\Delta_r - \frac{h}{L} \times$$

$$\times (\Delta_{12} + \Delta_{22}) + \frac{h}{L} (\Delta_{12} + \Delta_{14});$$

$$a_{2,p} = \Delta_{11} + \Delta_{13} - \Delta_{17} + 2\Delta_r + \frac{h}{L} (\Delta_{27} - \Delta_{47}) +$$

$$+ \frac{H+h}{L} (\Delta_{14} + \Delta_{24} - \Delta_{12} - \Delta_{22});$$

$$a_{3,p} = \Delta_{13} + \Delta_{11} - \Delta_{17} + 2\Delta_r +$$

$$+ \frac{H+h}{L} (\Delta_{27} - \Delta_{47}) +$$

$$+ \frac{2H+h}{L} (\Delta_{41} - \Delta_{21} + \Delta_{24} - \Delta_{22});$$

$$a_{4,p} = \Delta_{11} + \Delta_{13} - \Delta_{17} + 2\Delta_r + \frac{2H+h}{L} (\Delta_{27} - \Delta_{47}) +$$

$$+ \frac{3H+h}{L} (\Delta_{14} + \Delta_{24} - \Delta_{12} - \Delta_{22}).$$

We note that coefficient  $a_{2,1}$  is very small as compared with  $a_{1,1}$ . If we assume that  $a_{2,1} \approx 0$ , then the system breaks up and for the sought  $\beta_j$  we get

$$\beta_1 = \frac{a_{1,p}}{a_{1,1}}.$$

The forces arising on unit loading in the extreme end couplings will be denoted as  $a_j = 1 - \beta_j$ .

**Determination of forces  $X_j$  in lateral couplings.** After coefficients  $a_j$  are found, the difficulties involving the use of a statically indeterminate system vanish.

We now set up the canonical equations of the method of forces for the design schematic of Fig. 1a.

We denote the coefficients of these equations by  $\delta_{ik}$ . Each coefficient represents the sum

$$\delta_{ik} = \delta_{ik}^{(1)} + \delta_{ik}^{(2)} + \delta_{ik}^{(3)},$$

where  $\delta_{ik}^{(1)}$ ,  $\delta_{ik}^{(2)}$  and  $\delta_{ik}^{(3)}$  are, respectively, the translations due to deformations of panels, couplings, columns and the foundations.

We examine the determination of translations due to deformation of panels by reference to coefficient  $\delta_{12}^{(1)}$  (Fig. 5), which is the translation in direction 2-2 in state (I) due to force  $X_1 = 1$ .

If the translations of state (I) are regarded as feasible, then

$$\delta_{12}^{(i)} = - \sum_n W_{i-11}^{(n)},$$

where  $n$  is the number of the story.

The right-hand side of this equality is the work of the internal forces of state (II) on translations of state (I), summed over all the stories. The work of internal forces in each panel can be replaced by the work of external forces applied to it, taken with an opposite sign. Consequently, if we imagine that the couplings have been removed and replaced by the corresponding forces (which are denoted as  $Q_k^{(n)}$  for state (I) and by  $P_i^{(n)}$  for state (II), where  $i$  and  $k$  are numbers of couplings), then the work of internal forces within the limits of the  $n$ th story ( $W_{i-11}^{(n)}$ ) will be equal to the sum of works of forces  $P_1^{(n)}, P_2^{(n)}, \dots, P_i^{(n)}$  on translations of the points of application of these forces in state (I).

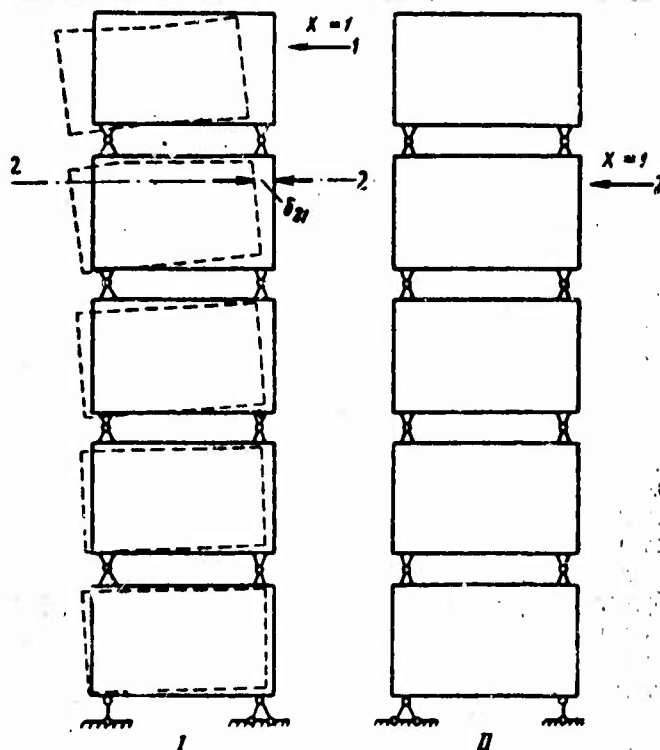


Fig. 5

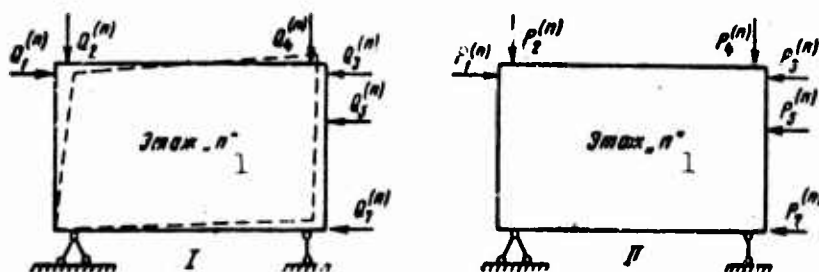


Fig. 6. 1)  $n$ th story.

Since the translation of the  $i$ th point of the panel in the  $i$ th direction for state (I) is  $\sum_{k=1}^7 \Delta_i Q_k^{(n)}$  (Fig. 6), then for the  $n$ th story we get

$$W_{i-11}^{(n)} = \sum_{l=1}^7 \sum_{k=1}^7 P_l^{(n)} Q_k^{(n)} \Delta_{ik}$$

and, summing over all the stories,

$$\delta_{12}^{(1)} = \sum_n W_{i-11}^{(n)} = \sum_{n=1}^5 \sum_{l=1}^7 \sum_{k=1}^7 P_l^{(n)} Q_k^{(n)} \Delta_{ik}.$$

Calculations using the formula thus obtained are quite cumbersome, for which reason use should be made of the system's regularity.

In calculations we have to consider panels of five stories for five-unit loads. However, there will be a total of nine different combinations. The five main combinations I-V are shown in Fig. 7. Due to the fact that panels of the 1st story have special support conditions, we have four additional combinations: I\*, II\*, III\* and IV\*, which differ only by the magnitude of horizontal forces in the bottom corners, i.e., in the left corner they are zero, while in the right corner they are equal to unity.

We denote the work of forces of combination  $m$  attendant to translations of combination  $l$  by  $w_{ml}$ ; it is determined in precisely the same manner as  $W_{i-11}^{(n)}$ . Then, as follows from Fig. 5:

$$\delta_{12}^{(1)} = W_{12} + W_{23} + W_{34} + W_{45}$$

and in the general case

$$\delta_{ik}^{(1)} = \sum_{j=1}^{5-k} W_{j, k-i-j} + W_{6-i, (6-k)*}.$$

The last term in this sum pertains to principal combination  $6-i$  and to the additional combination  $(6-k)*$ .

We now pass on to determination of translations  $\delta_{ik}^{(2)}$ , which are calculated as the sum of works of forces in the lateral ( $w_{ik}^{(6)}$ ) and the end ( $w_{ik}^{(7)}$ ) couplings

$$\delta_{ik}^{(2)} = w_{ik}^{(6)} + w_{ik}^{(7)}.$$

Here, obviously,

$$w_{ik}^{(6)} = \begin{cases} 0, & \text{when } i \neq k; \\ \Delta_6 & \text{when } i = k. \end{cases}$$

For the work of forces in end couplings, using typical combinations of forces shown in Fig. 7, we get the general expression

$$w_{ik}^{(n)} = \frac{4\Delta_{\kappa}}{L^3} \sum_{i=0}^{5-k} (jH+h) [(k-i+j)H+h] + \\ + 2\Delta_r \sum_{i=1}^{6-k} (\alpha_i \alpha_{k-i+1} + \beta_i \beta_{k-i+1}).$$

It remains to determine  $\delta_{ik}^{(3)}$ .

We denote by  $\Delta_{\kappa,r}$  ( $\Delta_{\kappa,v}$ ) the horizontal (vertical) translation of the top of the columns due to unit horizontal (vertical) load.

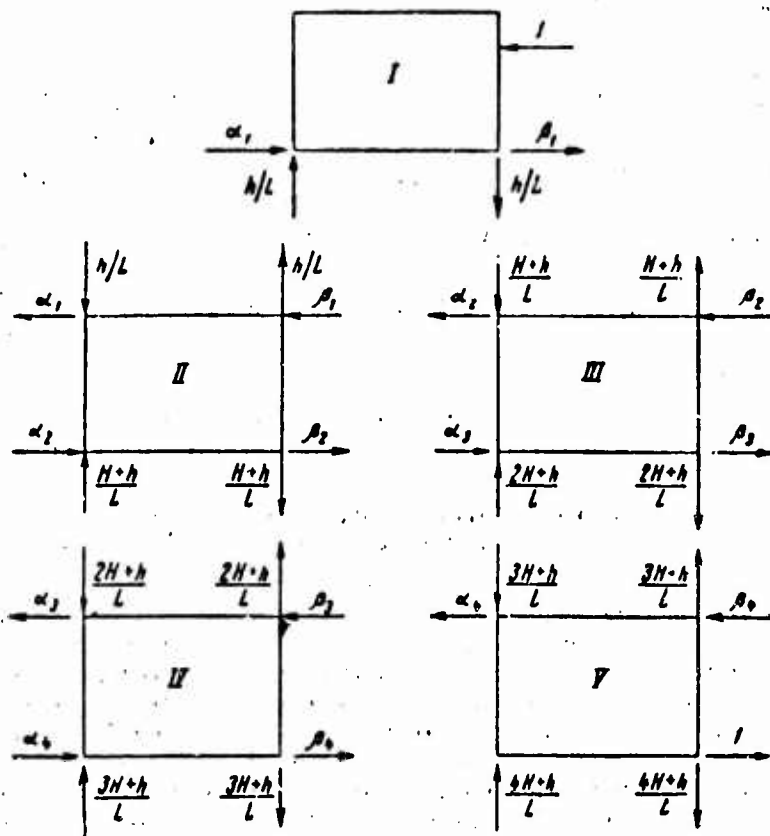


Fig. 7

Let the forces in the extreme columns in the  $i$ th state, i.e., on application of unit load  $X_i = 1$ , be  $R_i$ , while in the  $k$ th state let them be  $R_k$ . The forces in the center column will be twice as large, i.e.,  $2R_i$  and  $2R_k$ . For  $\delta_{ik}^{(3)}$  by analogy with the expression for  $W_{ik}^{(n)}$  we get

$$\delta_{ik}^{(3)} = \Delta_{\kappa,r} + \frac{1}{2} [R_i R_k + (2R_i)(2R_k) + R_i R_k] \Delta_{\kappa,v} = \Delta_{\kappa,r} + 3R_i R_k \Delta_{\kappa,v}.$$

Replacing  $R_i$  and  $R_k$  by the expressions shown in Fig. 7, we find

$$\delta_{ik}^{(3)} = \Delta_{k,r} + 3\Delta_{k,b} \frac{(5-k)H+h}{L} \frac{(5-l)H+h}{L}$$

The elastic compliance of the foundation may be taken into account when determining  $\Delta_{k,r}$  and  $\Delta_{k,b}$ . The nonelastic settling of the ground is approximately assumed to be independent of the load; it is contained in the free terms.

**Free terms of canonical equations.** a) *Panels loaded symmetrically along the bottom flange.*

We shall assume that the translations  $\Delta_{iq}$  of points on the contour of a freely supported panel due to a uniformly distributed load are known.

The sought load terms are expressed by the sum

$$\delta_{iq} = \delta_{iq}^{(a)} + \delta_{iq}^{(b)} + \delta_{iq}^{(n)} + \delta_{iq}^{(r)}$$

where  $\delta_{iq}^{(a)}$  are the translations of the  $i$ th point of the contour of the panel in the  $i$ th direction due to the load exerted by the given story;

$\delta_{iq}^{(b)}$  are the same, due to the load exerted by above-lying stories;

$\delta_{iq}^{(n)}$  are the same, due to deformation of couplings;

$\delta_{iq}^{(r)}$  are the same, due to deformation of column and elastic deformations of the ground.

The translations are found entirely by analogy with determination of coefficients  $\delta_{ik}$ . For symmetrical panels  $\delta_{iq}^{(b)}$  vanish. The term  $\delta_{iq}^{(a)}$  may be usually disregarded due to the smallness of forces in the horizontal end couplings.

b) *Nonuniform settling of supports.* Denoting the difference in settling of extreme and central supports by  $\Delta_0$  and assuming that the center support settles more, we get

$$\delta_{i0} = -\frac{(5-i)H+h}{L} \Delta_0$$

## §2. Experimental Determination of the Stiffness of Panels and Rigidity of Couplings

**Stiffness of panels.** As was previously mentioned, nonelastic deformations are, as a rule, not considered in the design of panels. To estimate the errors which arise as a result of this, we have tested full-size panels with and without openings. Concentrated forces were applied at points 1-7 (Fig. 3); translations  $\Delta_{ik}$  were measured by indicators. In addition, the bottom flanges of the panels were subjected to a uniformly distributed load. The over-all view of the testing setup is shown in Fig. 8, while the results of experiments for a panel with an opening, analyzed on the assumption of mutuality of translations are tabulated in Table 1.

TABLE 1

Values of  $\Delta_{ik}$  and  $\Delta_{iq}$  in Microns Obtained by Measurement on Full-Size Panels

$i \backslash k$	1	2	3	4	5	6	7	$q=1 \tau/\mu$ A
1	508	-27	-194	3	-410	-55	-193	829
2	-27	66	88	0	-1	0	-1	72
3	-194	88	344	3	90	0	116	-439
4	-3	0	3	44	-5	39	3	37
5	-410	-1	90	-5	688	0	152	-702
6	-55	0	0	39	0	39	0	30
7	-193	-1	116	3	152	0	310	-53

A)  $t/m$ .

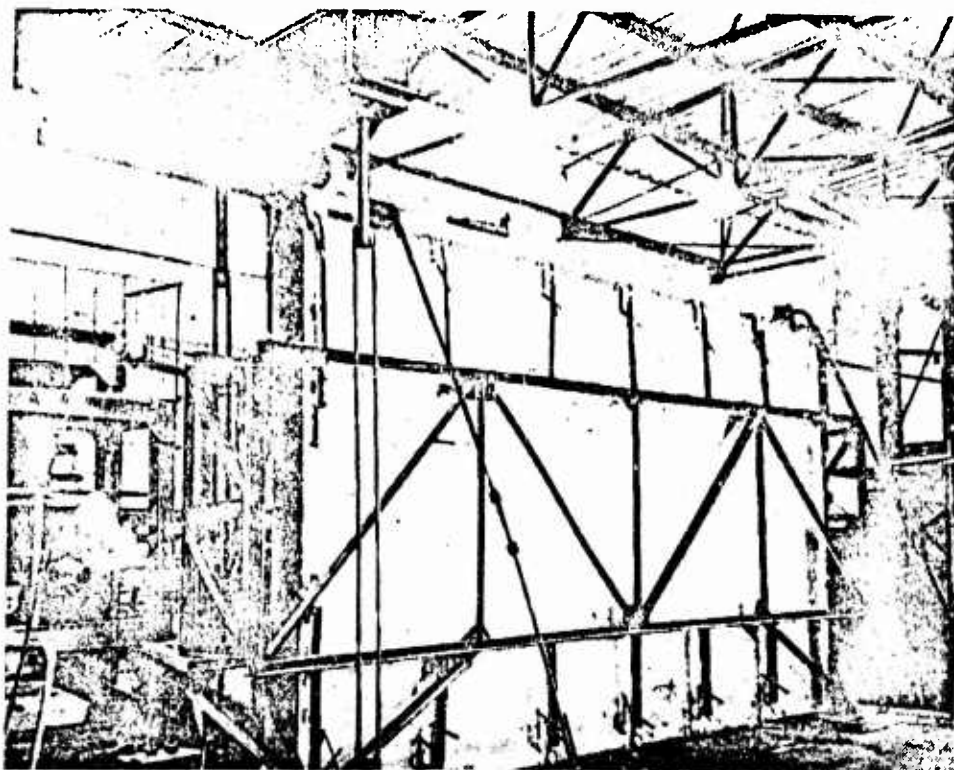


Fig. 8 GRAPHIC NOT REPRODUCIBLE

The values of  $\Delta_{ik}$  and  $\Delta_{iq}$  thus obtained could have been compared with those obtained by computation. However, due to the complex shape of the panels, a sufficiently accurate calculation for them requires laborious numerical methods. Hence the magnitudes of translations  $\Delta_{ik}$  and  $\Delta_{iq}$  on the assumption of an elastically functioning material were obtained not by computation, but by testing elastic models made to the 1:8 scale from organic glass.

The over-all view of the model and of the testing setup is shown in the photograph of Fig. 9. Similarity between the model and original was given by the formula

$$\Delta_o = \Delta_n \frac{E_m}{E_o} k_1 k_2,$$

where:  $\Delta_o$  are the translations of points on the contour of the original;  
 $\Delta_m$  are the translations of points on the contour of the model;  
 $E_o$  is the elastic modulus of the material of the original;  
 $E_m$  is the elastic modulus of the material of the model;  
 $k_1$  is the ratio of the dimensions of the model to those of the original;  
 $k_2$  is the ratio of the load applied to the original to the load applied to the model.

The results thus obtained are tabulated in Table 2.

TABLE 2

Values of  $\Delta_{ik}$  and  $\Delta_{iq}$  in Microns, Obtained by Testing Elastic Models

$k \backslash i$	1	2	3	4	5	6	7	$q=1 \text{ } t/m$ A
1	158	15	-79	-17	-128	-17	-60	267
2	15	66	28	-4	21	0	0	22
3	-79	28	107	-21	128	0	49	-152
4	-17	-4	-21	54	-26	45	0	+55
5	-128	21	128	-26	214	0	47	-260
6	-17	0	0	45	0	45	0	4
7	-60	0	49	0	47	0	96	0

A)  $t/m$ .

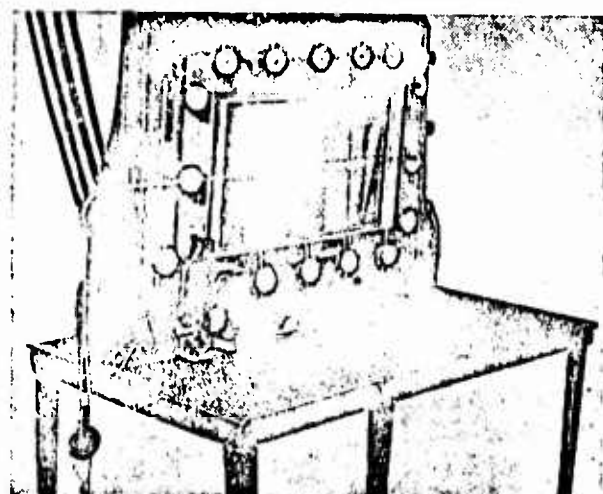
As is shown by comparing these data with those presented in Table 1, the ratio of the actual translations to those of a perfectly elastic panel (this ratio will be denoted by  $\gamma$ ) is usually greater than unity. The value of  $\gamma$  changes when a unit load is moved along the panel's contour. If the vertical load is applied to the upper corner, then  $\gamma \approx 1$ , if it is applied to the upper flange in the  $\frac{1}{4}$  point of the span, then  $\gamma = 1.23$ , while for a mid-span loading  $\gamma = 2.1$ . The corresponding values for the bottom flange are  $\gamma = 1, 1.64, 2.13$ . For a horizontal load  $\gamma$  varies from 2 to 2.5.

Thus, depending on whether compressive, tensile or shear strains predominate, the actual stiffness may be found to be the same as for a perfectly rigid panel, but also may be by a factor of 1.5-2.5 lower. It is natural that this circumstance must be considered in design.

The causes of the reduced stiffness of full-size panels should be sought in the presence of production-process cracks and elevated deformability of concrete in tension.

**Rigidity of couplings.** All the couplings between panels (Fig. 2) are implemented by means of insertable connections of a single

type in the form of a "table," consisting of a steel plate and anchors from smooth or shaped round steel. Since the panels are connected by several insertable connections situated at quite large distances from one another, then it may be assumed that each panel resists only axial forces, as this is shown in Fig. 10a.



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Fig. 9

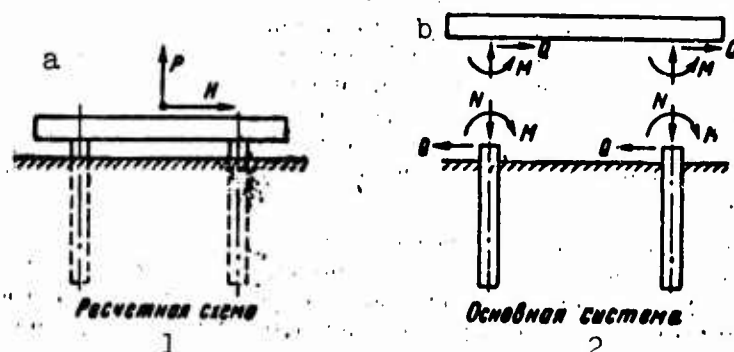


Fig. 10. 1) Principal scheme; 2) main scheme.

Connections using insertable components are inevitably compliant. The main source of compliance is the insertion of anchors in the concrete. The anchors of the insertable part, as shown in Fig. 10b, are subjected to axial forces ( $N$ ), shear forces ( $Q$ ) and bending moment ( $M$ ). The anchors are usually made from a periodic-shape steel from 10 to 16 mm in diameter. In order to characterize the compliance of the insertion of anchors, Fig. 11 shows experimental curves of the axial translation ( $g$ ) as a function of  $N$ , lateral translation ( $\delta$ ) as a function of  $Q$  and the angle of twist ( $\phi$ ) as a function of  $M$  for an anchor 14 mm in diameter, inserted into 200 concrete to a depth of 20 cm. It follows from these graphs that:

a) the compliance of anchor insertion increases nonlinearly with an increase in load;

b) for stresses in anchors equal to the design strength of

steel (the anchors are made from 25Г2С steel) the axial translations ( $g$ ) are as high as 170  $\mu\text{m}$ , i.e., they are commensurable with the translations of points on the contours panels. The lateral translations ( $\delta$ ) are even greater; for the same forces they exceed the axial translations by approximately an order of magnitude.

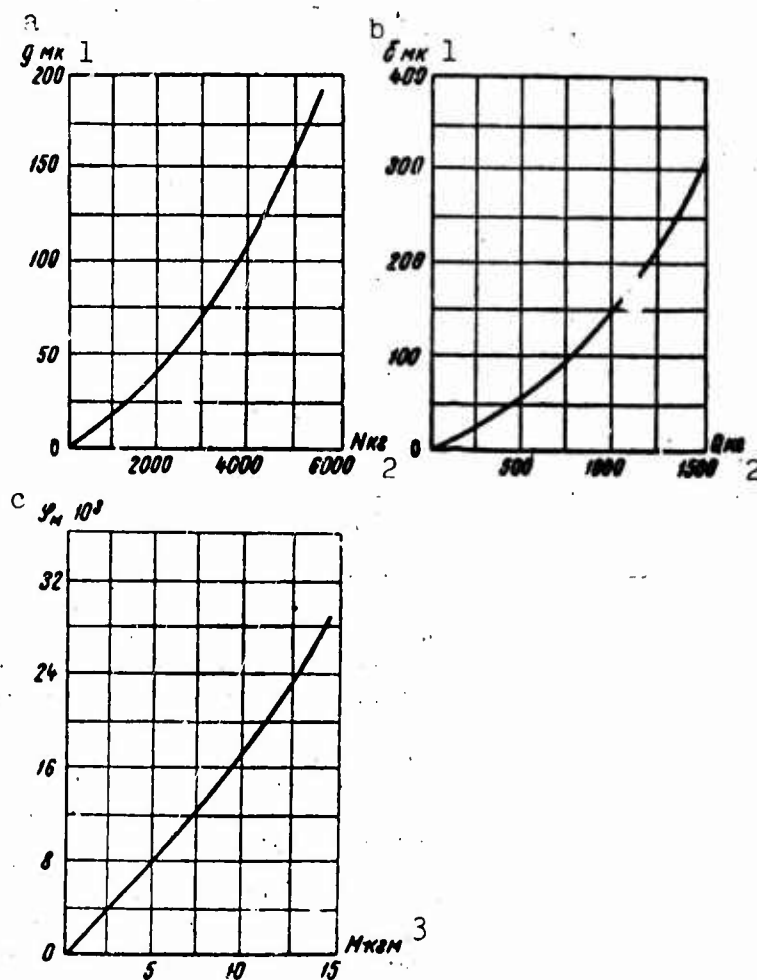


Fig. 11. 1)  $\mu\text{m}$ ; 2)  $\text{kg}$ ; 3)  $\text{kg-m}$ .

Consequently, if we want to know the actual force distribution between structural elements, it is at least as erroneous to consider the couplings as perfectly rigid as imparting these properties to panels. The latter, it should be said, is almost never done, while the first is assumed by far not always.

Apparently, it is completely expedient to limit the deformations of couplings by limits within which they may be approximately regarded as linearly deformable. It is assumed henceforth that the deformations do not go past these limits.

### §3. Results of Calculating the Transverse Structure and Estimate of the Effect of the Stiffness of its Elements

In accordance with the results presented in §1, we have designed the transverse structure for the combined effect of a sym-

metrical vertical load and nonuniform settling of supports. In these calculations we assumed:

a) the difference between the settling of the extreme and center supports is

$$\Delta_0 = 0,0015L = 7,32 \text{ mm};$$

b) the compliances of the couplings are

$$\Delta_r = 60 \text{ } \mu\text{m/t}; \quad \Delta_s = 7 \text{ } \mu\text{m/t}; \quad \Delta_0 = 30 \text{ } \mu\text{m/t}; \quad \Delta_R = 0.$$

The stiffness of the panels was determined from Table 1.

The results of these calculations are shown in Fig. 12.

The interaction between the left and right sides of the transverse structure are characterized by forces in lateral couplings which vary with the height; the presence of these forces overloads the side couplings of the first story, and also of the extreme supports as compared with the center supports (42.66 and 17.34 t).

If one uses, instead of the data of Table 1, the stiffnesses of panels determined by testing elastic models, then, as was shown by calculations, the forces in the end couplings will be found to be greater than given in Fig. 12 by approximately 20%, while the forces in the extreme supports will now be not 2-, but 3-fold greater than in the middle support. This again shows the need of a sufficiently accurate estimate of the stiffness of a building's elements. On the other hand, the problem is raised whether it is possible, within the permissible limits of stiffness of couplings and panels, to obtain an optimal distribution of forces between elements. Such an analysis was made.

For the structure under consideration it was found to be practically possible to vary the stiffness characteristics within the following limits:

a) for panels - from the values in Table 1, corresponding to panels from heavy concrete, to a stiffness 1.5-3-fold smaller when using light concretes;

b) for side couplings to 500  $\mu\text{m/t}$ ;

c) for horizontal end couplings to 150  $\mu\text{m/t}$ ;

d) for vertical end couplings to 20  $\mu\text{m/t}$ .

As a result of the system's regularity, it is most desirable to achieve a uniform distribution of forces among its element; the following parameters were taken as uniformity criteria:

a) maximum force in lateral couplings -  $X_{\text{max}}$ ;

b) maximum force in horizontal end couplings -  $H$ ;

c) quantity  $n$ , being the ratio of the total force in the extreme supports of the 1st story to half the vertical load.

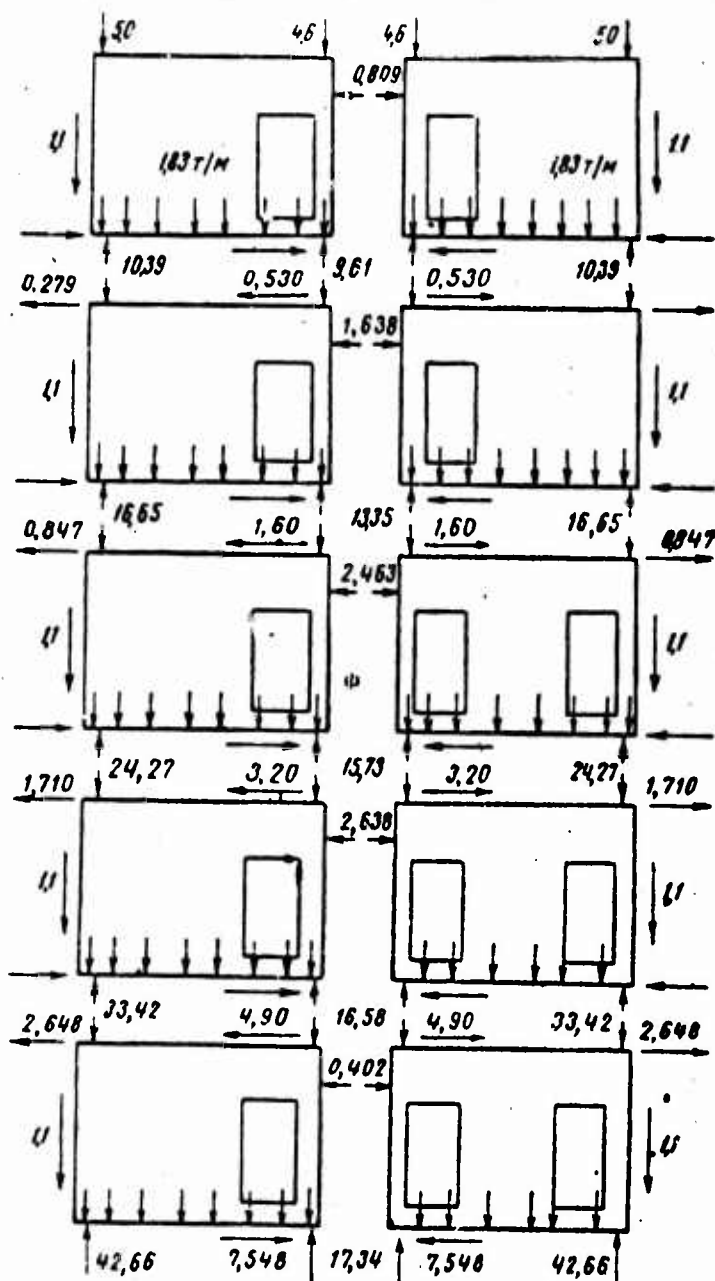


Fig. 12

The calculations were made on the assumption that the columns are perfectly rigid, and that the ground is not subject to elastic deformations. Coefficient  $k$  expresses the ratio of the actual stiffness of a panel to that calculated on the basis of the elastic properties of materials.

The results of calculations are tabulated in Table 3. The main conclusion following from Table 3 consists in the beneficial effect of reducing the stiffness of the panels, as well as the rigidity of couplings. The character of this effect is such that, with moderate reservations, it is possible to regard the effect of coupling rigidity as a local phenomenon. Thus, increasing the compliance of lateral couplings we first of all reduce the forces

in these couplings; a similar effect is achieved by reducing the rigidity of end couplings in the horizontal direction. From the point of view of equalizing the pressures on the supports it is most effective to reduce the stiffness of panels; however, it is apparently impossible to obtain in practice a value of  $n$  below 1.4.

TABLE 3

1 Характеристика распределения усилий	k=1			k=1			k=1			$\Delta_r=60, \Delta_B=7, \Delta_G=30$		
	Жесткость боковой связи			Жесткость вертикальной торцевой связи			Жесткость горизонтальной торцевой связи			Жесткость панели		
	2	3	4	5	6	7	8	9	10	11	12	13
	$\Delta_G=0$	$\Delta_G=100$	$\Delta_G=500$	$\Delta_B=0$	$\Delta_B=10$	$\Delta_B=20$	$\Delta_r=0$	$\Delta_r=75$	$\Delta_r=150$	$k=1$	$k=2,5$	$k=5$
H	21,76	19,99	16,06	21,76	19,58	18,00	21,76	12,1	8,37	12,5	7,12	4,17
$X_{\max}$	7,64	5,86	3,57	7,64	7,4	7,108	7,64	4,1	2,82	4,31	2,64	1,58
n	2,17	2,15	2,03	2,17	1,96	1,83	2,17	19,7	1,83	18,7	15,1	13,3

- 1) Force distribution characteristic
- 2) Rigidity of lateral coupling
- 3) Rigidity of vertical end coupling
- 4) Rigidity of horizontal end coupling
- 5) Panel stiffness.

TABLE 4

1 Характеристики	2 Жесткость связей		
	$\Delta_r=0$ $\Delta_G=0$ $\Delta_B=0$	$\Delta_r=75$ $\Delta_G=100$ $\Delta_B=10$	$\Delta_r=150$ $\Delta_G=500$ $\Delta_B=20$
H	21,76	11,06	7,27
$X_{\max}$	7,64	3,33	1,89
n	2,00	1,77	1,57

- 1) Characteristics; 2) rigidity of couplings.

Increasing the compliance of couplings above these limits may result in loosening of the seams; a further reduction in the stiffness of panels is difficult to obtain. Hence one should strive to a simultaneous reduction of the rigidity of all the structural elements.

It is clear that this conclusion is valid only from the point of view of the effect on nonuniform settling of supports, while the decisive factor for the action of the load is the ratio of the rigidities, rather than their absolute value. The effect of simultaneous variation of the rigidity of all the couplings is characterized by Table 4.

A reduction in the rigidity of couplings within feasible limits thus, first of all, results in a sharp reduction of horizontal forces;  $X_{\max}$  is reduced approximately 4-fold, while  $H$  drops ap-

proximately 3-fold. The vertical forces in end couplings due to the vertical load remain almost unchanged, and those due to settling of supports become substantially smaller in the lower stories.

We note that we are dealing with a simultaneous change in rigidities of all the couplings each time when the strength of the concrete of the panel into which the coupling anchors are inserted is changed.

The above calculations were made on the assumption that the columns are perfectly rigid and with consideration only of the inelastic settling of the supports. To estimate the possible effect of elasticity of columns and of the foundations, the transverse structure was designed for three values of  $\Delta_k$ : 8  $\mu\text{m/t}$ , 70  $\mu\text{m/t}$  and 230  $\mu\text{m/t}$ , which corresponds to a transition from a rock foundation to the weakest soils encountered in practice. The calculations were made on assumption of perfectly rigid couplings and with  $k = 1$ .

The results are presented in Table 5 separately for vertical load and settling of supports, since the effect of  $\Delta_k$  was found to be different for these two effects.

TABLE 5

1 Характеристики	2 Вертикальная нагрузка			3 Осадка опор			4 Суммарное влияние нагрузки и осадки опор		
	5 $\Delta_k$ в микронах								
	8	70	230	8	70	230	8	70	230
$H$	2,35	3,75	4,98	10,16	6,54	3,38	12,51	10,29	8,36
$X_{\max}$	1,15	1,47	1,88	3,52	2,22	1,09	4,67	3,69	2,97
$n$	1,14	1,24	1,33	—	—	—	1,87	1,71	1,57

- |                         |  |
|-------------------------|--|
| 1) Characteristics      | 4) Total effect of load and settling of supports |
| 2) Vertical load        | 5) In microns.                                   |
| 3) Settling of supports |  |

With an increase in  $\Delta_k$  the horizontal forces  $H$  and  $X_{\text{max}}$  due to the vertical load increase, while those due to inelastic settling of supports decrease. As a whole, the effect of reducing the elastic stiffness of the foundation for the given value of  $\Delta_0$  was found to be positive. It is desirable to simultaneously reduce the inelastic and increase the elastic deformation of ground. The positive effect of elastic deformations of the foundation, due to which the forces produced by settling of the foundation are redistributed, nevertheless by no means allows to expect complete elimination of nonuniformity in the distribution of forces among the building's elements.

The analysis which was made makes it possible to note certain features in the functioning of the transverse structure:

a) the forces produced in lateral couplings by the vertical load increase downward, while those due to settling of the center support, conversely, increase upward. The design forces are the forces in the couplings of the second story;

b) the vertical force in end couplings reaches its maximum between the columns and panels of the first story;

c) the horizontal force in the end couplings reaches its maximum in the extreme couplings between the first and second stories and in the center couplings between the columns and panels of the first story.

#### §4. Conclusions

1. Modern typical structures of large-panel residential buildings are produced by the hundreds and thousands. It is hence expedient, in order to refine starting data for design, to have resort to experiments, as this is done, for example, in the aircraft industry. It is possible that after accumulating experimental data it will be possible to replace tests of full-size elements by testing models or by calculations.

2. Tests of full-size panels and of elastic models have shown that, depending on the kind of predominant deformation, the actual stiffness of panels may be found almost the same as in an ideal panel, but may also be smaller by a factor of 1.5-2.5.

The compliance of couplings is of the same order of magnitude as the translations of points on the panel contours.

3. The direct application of the principle of feasible translations makes it possible to perform calculations with an accurate consideration of the actual stiffness of panels and rigidity of couplings.

The design of the transverse structure of a series K-7 building, made on the basis of experimental data, has shown that the above method is practically feasible. Despite the relative complexity of the structure, it was possible to study the effect of the majority of major factors on its functioning.

The studies were performed in the All-Union Scientific Research Institute of Reinforced-Concrete Products and Nonmetallic Materials.

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#### Transliterated Symbols

187	r = g = gorizontal'nyy = horizontal
187	b = v = vertikal'nyy = vertical
187	б = b = bokovoy = lateral
192	т = t = tortsovoy = end
192	к.г = k.g = kolonna, gorizontal'noye = column, horizontal
192	к.в = k.v = kolonna, vertikal'noye = column, vertical
195	о = o = original = original
195	м = m = model' = model
198	к = k = kolonna = column

## DESIGN OF THREE-DIMENSIONAL COMPOSITE AND MONOLITHIC THIN-WALLED BOX-TYPE STRUCTURES

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The present article gives a derivation of systems of differential equations describing the stressed-strained state of composite and monolithic thin-walled box-type structures subjected to a three-dimensional load and for different conditions of connecting the end facets-diaphragms to the longitudinal facets-plates. Equations for composite structures, whose longitudinal facets are interconnected by four types of couplings were assumed on the assumption that only the longitudinal shear couplings are elastocompliant. Integration of the obtained systems of equations for certain kinds of loads, most frequently encountered in practice, is performed in the closed form.

The solutions of the present articles are based on equations obtained in [2] for the study of composite thin-walled, multi-jointed prismatic structures.

### §1. The Stressed-Strained State of Box-Type Structures Subjected to Three-Dimensional Loading

We consider a box-type structure, composed of four plates interconnected in its longitudinal seams by four types of couplings (Fig. 1), with the couplings preventing transverse translation and mutual rotation of the edges of facets being regarded as perfectly rigid, while the couplings resisting longitudinal shear being considered as elastocompliant, with a finite rigidity  $\beta$  and permitting mutual translation of the edges of the facets in the longitudinal direction. This box structure is henceforth called a composite box structure, as opposed to a monolithic structure, in which all the couplings joining the individual plates are perfectly rigid. The dimensions  $h_g$ ,  $h_c$  and  $h_u$  define, respectively, the positions of the centers of gravity, of flexure and of torsion (Fig. 1).

Due to the fact that the transverse and angular couplings are perfectly rigid and also due to the smallness of the twisting moments, which can be taken up by individual facets, Eq. (10) obtained in [2] will be written in the form

$$\left. \begin{aligned} \sum \sum E a_{ij,ry} U'_{ij}(z) - \sum \sum \beta_t \bar{a}_{ij,ry} U_{ij}(z) - \sum \sum G b_{ij,ry} U_{ij}(z) - \\ - \sum \sum G c_{kl,ry} V'_{kl}(z) + p_{ry} = 0; \\ \sum \sum G c_{kl,ry} U'_{ij}(z) + \sum \sum G r_{kl,ry} V'_{kl}(z) - \sum \sum s_{kl,ry} V_{kl}(z) + q_{ry} = 0, \end{aligned} \right\} \quad (1)$$

where

$$\left. \begin{aligned} a_{ij,ry} &= \int_F \varphi_{ij}(s) \varphi_{ry}(s) dF; \\ a_{ij,ry} &= \sum_i [\varphi'_{ij}(\bar{s}) - \varphi'^{i+1}_{ij}(\bar{s})] [\varphi'_{ry}(\bar{s}) - \varphi'^{i+1}_{ry}(\bar{s})]; \\ b_{ij,ry} &= \int_F \varphi'_{ij}(s) \varphi'_{ry}(s) dF; \quad c_{kl,ry} = \int_F \psi_{kl}(s) \varphi'_{ry}(s) dF; \\ c_{ij,ry} &= \int_F \varphi'_{ij}(s) \psi_{ry}(s) dF; \quad r_{kl,ry} = \int_F \psi_{kl}(s) \psi_{ry}(s) dF; \\ s_{kl,ry} &= \int_L \frac{M_{kl} M_{ry}}{EJ} ds; \quad p_{ry}(z) = \int_L p \varphi_{ry}(s) ds; \quad q_{ry}(z) = \int_L q \psi_{ry}(s) ds; \end{aligned} \right\} \quad (2)$$

$U_{ij}(z)$  and  $V_{kl}(z)$  are the sought generalized translations of the middle surface of an individual component of the plate;

$\varphi_{ij}(s)$  and  $\psi_{kl}(s)$  are functions of the transverse distribution of translations, which are to be preselected;

$E$  and  $G$  are the 1st and 2nd kind moduli of elasticity of the plate material;

$\beta_t$  is the coefficient of rigidity of longitudinal shear couplings of the  $t$ th seam;

$p = p(z; s)$  and  $q = q(z; s)$  are specified external forces.

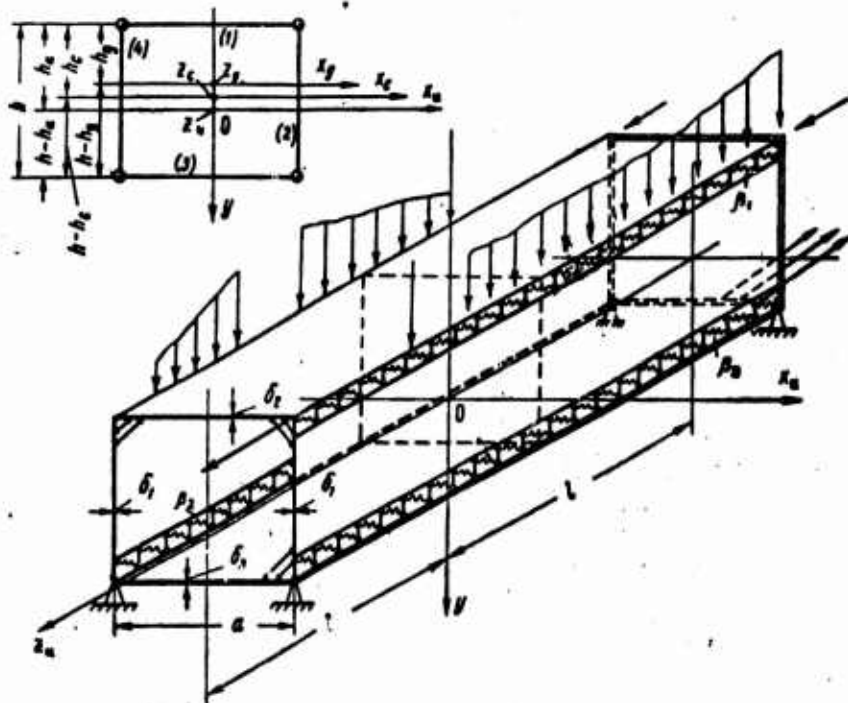


Fig. 1

In the absence of angular couplings between the plates making up the box structure (the longitudinal seams are cylindrical hinges) the third term in the second equation of System (1), expressing the work on internal forces on flexural deformations of the strip in its plate will vanish.

We cut out from the structure an elementary strip of width  $dz$ . Since each element of the strip has two degrees of freedom with respect to longitudinal translations, and the entire strip as a whole has four degrees of freedom in the plane of the cross section (taking into account deformation of the contour), then the total number of degrees of freedom of an elementary strip is

$$A = \sum_{j=1}^4 m_j + \sum_{j=1}^4 n_j = 2 \cdot 4 + 4 = 12,$$

where  $m_j$  and  $n_j$  are the number of degrees of freedom with respect to the longitudinal and transverse translations of an elementary strip cut out from the  $j$ th element of the structure.

Henceforth we assume the following limits of summation in calculating individual terms of equations

$$j=4; i=2; k=4.$$

The number of functions of the transverse distribution of translations should thus be 12 and the components of the vector of the total translation of an arbitrary point of the structure will be written as

$$\left. \begin{aligned} u(z; s) &= \sum_{j=1}^4 \sum_{i=1}^2 U_{ij}(z) \varphi_{ij}(s) = U_{11}(z) \varphi_{11}(s) + U_{12}(z) \varphi_{12}(s) + \\ &\quad + \dots + U_{24}(z) \varphi_{24}(s); \\ v(z; s) &= \sum_{j=1}^4 \sum_{k=1}^2 V_{kj}(z) \psi_{kj}(s) = V_{1,1-4}(z) \psi_{1,1-4}(s) + \\ &\quad + \dots + V_{4,1-4}(z) \psi_{4,1-4}(s). \end{aligned} \right\} \quad (3)$$

Functions  $\varphi_{ij}(s)$  are selected in the form

$$\begin{aligned} \varphi_{11}(s) &= 1; \varphi_{12}(s) = 1; \varphi_{13}(s) = 1; \varphi_{14}(s) = 1; \\ \varphi_{21}(s) &= x(s); \varphi_{22}(s) = y(s); \varphi_{23}(s) = x(s); \varphi_{24}(s) = y(s), \end{aligned}$$

where  $x(s)$  and  $y(s)$  are coordinates of an arbitrary point on the contour, while functions  $\psi_{kj}(s)$  are selected in the form

$$\begin{aligned} \psi_{1,1-4}(s) &= h(s); \psi_{2,1-4}(s) = x'(s); \\ \psi_{3,1-4}(s) &= y'(s); \psi_{4,1-4}(s) = x'(s)y(s) + x(s)y'(s), \end{aligned}$$

where  $h(s)$  is the distance along a normal from center  $O$  to a corresponding plate facet,  $x'(s)$ ,  $y'(s)$  are derivatives of these coordinates with respect to  $s$ .

The diagrams of functions  $\varphi_{ij}(s)$ ,  $\psi_{kj}(s)$  and of derivatives  $\psi'_{ij}(s)$  are shown in Fig. 2.

The generalized translations for selected functions of the transverse distribution have a completely defined physical meaning.

From now on we denote

$$U_{ij}(z) = U_{ij} \text{ and } V_{kj}(z) = V_{kj}.$$

We now show for illustration how the expansion of the second term of the first of equations of System (1) is written for  $r_{ij} = 11$ . We shall traverse the contour of the elementary strip clockwise, i.e., the values of the mutual longitudinal translation of the two elements being connected will be determined in the point of contact by subtracting from the values of function  $\varphi_{ij}(s)$  for elements (1), (2), (3), (4) of the strip the values of this same function, respectively, for elements (2), (3), (4), (1) (see Fig. 1). Thus

$$\sum_{j=1}^4 \sum_{i=1}^2 \beta_i \bar{a}_{ij,11} U_{ij} = \beta_1 (\bar{a}_{11,11} U_{11} + \bar{a}_{12,11} U_{12} + \dots + \bar{a}_{24,11} U_{24}),$$

where

$$\begin{aligned} \beta_1 \bar{a}_{11,11} &= \beta_1 \sum [\varphi_{11}(s) - \varphi_{11}^{l+1}(s)] [\varphi_{11}(s) - \varphi_{11}^{l+1}(s)] = \beta_1 [0-1][0-1] + \\ &+ \beta_1 [1-0][1-0] + \beta_2^0 + \beta_2^0 = \beta_1 + \beta_1 = 2\beta_1, \\ &\text{since } \beta_2^0 = \beta_2 [0-0][0-0] = 0. \end{aligned}$$

The remaining coefficients of the unknown functions of other terms of the equations are determined similarly.

Upon performing the required computations, we get a system of differential equations which define the stressed-strained state of a composite box-like structure subjected to a three-dimensional load. The system in matrix form is shown in Table 1. The coefficients of the matrix are reciprocal, and the matrix proper is symmetrical about the principal diagonal.

Certain particular cases, to which it is possible to pass on from the system of equations obtained, are of interest.

Thus, assuming  $\delta_2 = \delta_3$ ,  $h_g = h_u = h_c = \frac{h}{2}$  and  $\beta_1 = \beta_2$ , we get a system of equations which define the stressed-strained state of a composite box-type structure whose cross section has two axes of symmetry.

If all the couplings in the seams are assumed to be perfectly rigid, thus eliminating mutual translations of the component plate-facets of the box structure, then the coefficients of the unknown functions in the matrix of Table 1 containing  $\beta_1$  and  $\beta_2$  will be zero, and the system of equations thus obtained will describe the stressed-strained state of a monolithic box structure which has the same cross section as corresponding composite structure.

It is natural that the system of equations presented on page 283 of [1] for a monolithic box structure is a particular case of the system presented in Table 1.

TABLE 1

Matrix of the System of Equations, Describing the Stressed-Strained State of a Composite Box-Type Structure (the Cross Section has a Single Vertical Axis of Symmetry) Subjected to a Three-Dimensional Load

№ урав- нений A	$U_{11}(z)$	$U_{12}(z)$	$U_{13}(z)$	$U_{14}(z)$	$U_{21}(z)$	$U_{22}(z)$
1	$\dot{a}_{22} D^2 - a_{11}$	$a_{12}$	0	$a_{14}$	0	$-a_{16}$
2	$a_{21}$	$\dot{a}_{22} D^2 - a_{22}$	$a_{23}$	0	$a_{25}$	$\dot{a}_{26} D^2 + a_{26}$
3	0	$a_{32}$	$\dot{a}_{33} D^2 - a_{33}$	$a_{34}$	0	$a_{36}$
4	$a_{41}$	0	$a_{43}$	$\dot{a}_{44} D^2 - a_{44}$	$-a_{45}$	$\dot{a}_{46} D^2 + a_{46}$
5	0	$a_{52}$	0	$-a_{54}$	$\dot{a}_{55} D^2 - a_{55}$	0
6	$-a_{61}$	$\dot{a}_{62} D^2 + a_{62}$	$a_{63}$	$\dot{a}_{64} D^2 + a_{64}$	0	$\dot{a}_{66} D^2 - a_{66}$
7	0	$-a_{72}$	0	$a_{74}$	$\dot{a}_{75} D^2 + a_{75}$	0
8	0	$\dot{a}_{82} D^2 + a_{82}$	0	$-\dot{a}_{84} D^2 - a_{84}$	$-a_{85}$	0
9	0	0	0	0	$\dot{a}_{95} D$	0
10	0	0	0	0	$\dot{a}_{10,5} D$	0
11	0	0	0	0	0	$\dot{a}_{11,6} D$
12	0	0	0	0	$-\dot{a}_{12,5} D$	0
<div style="display: flex; justify-content: space-between;"> <div style="width: 30%;"> <math>\dot{a}_{11} = EF_2;</math>  <math>a_{11} = 2\beta_1;</math>  <math>a_{12} = a_{21} = \beta_1;</math>  <math>a_{14} = a_{41} = \beta_1;</math>  <math>a_{16} = a_{61} = 2h_c \beta_1;</math>  <math>\dot{a}_{22} = EF_1;</math>  <math>a_{23} = \beta_1 + \beta_2;</math>  <math>a_{25} = a_{52} = \frac{a}{2} (\beta_1 + \beta_2);</math>  <math>\dot{a}_{26} = \dot{a}_{62} = \frac{EF_1}{2} (h - 2h_c);</math>    Оператор: <math>\dot{\phantom{x}}</math>  <math>D = (\dots)</math>  <math>D^2 = (\dots)</math> </div> <div style="width: 30%;"> <math>a_{33} = a_{66} = \beta_1 h_c - \beta_2 (h - h_c);</math>  <math>a_{37} = a_{73} = \frac{a}{2} [\beta_1 h_g - \beta_2 (h - h_g)];</math>  <math>\dot{a}_{28} = \dot{a}_{82} = \frac{EF_1 a}{4} (h - 2h_g);</math>  <math>a_{36} = a_{63} = \frac{a}{2} [\beta_1 h_g - \beta_2 (h - h_g)];</math>  <math>\dot{a}_{33} = EF_2;</math>  <math>a_{35} = 2\beta_1;</math>  <math>a_{34} = a_{43} = \beta_1;</math>  <math>a_{38} = a_{83} = 2\beta_2 (h - h_c);</math>  <math>\dot{a}_{44} = EF_1;</math>  <math>a_{46} = \beta_1 + \beta_2</math> </div> <div style="width: 30%;"> <math>a_{45} = a_{54} = \frac{a}{2} (\beta_1 + \beta_2);</math>  <math>\dot{a}_{46} = \dot{a}_{64} = \frac{EF_1}{2} (h - 2h_c);</math>  <math>a_{48} = a_{84} = \beta_1 h_c - \beta_2 (h - h_c);</math>  <math>a_{47} = a_{74} = \frac{a}{2} [\beta_1 h_g - \beta_2 (h - h_g)];</math>  <math>\dot{a}_{48} = \dot{a}_{84} = \frac{EF_1 a}{4} (h - 2h_g);</math>  <math>a_{46} = a_{64} = \frac{a}{2} [\beta_1 h_g - \beta_2 (h - h_g)];</math>  <math>\dot{a}_{55} = \frac{Ea^3}{12} (F_2 + F_3);</math>  <math>a_{58} = \frac{a^3}{2} (\beta_1 + \beta_2) + G(F_2 + F_3);</math>  <math>\dot{a}_{57} = \dot{a}_{75} = \frac{Ea^3}{12} [-F_2 h_g + F_3 (h - h_g)];</math>  <math>a_{57} = a_{75} = \frac{a^3}{2} [\beta_1 h_g - \beta_2 (h - h_g)] + G[F_2 h_g - F_3 (h - h_g)]</math> </div> </div>						

TABLE 1 (continued)

$U_{11}(z)$	$U_{11}(z)$	$V_{1,1-4}(z)$	$V_{2,1-4}(z)$	$V_{3,1-4}(z)$	$V_{4,1-4}(z)$	Свободные члены В
0	0	0	0	0	0	$-p_{11}$
$-a_{27}$	$\dot{a}_{28} D^2 + a_{28}$	0	0	0	0	$-p_{12}$
0	0	0	0	0	0	$-p_{13}$
$a_{47}$	$-\dot{a}_{48} D^2 - a_{48}$	0	0	0	0	$-p_{14}$
$\dot{a}_{57} D^2 + a_{57}$	$-a_{68}$	$-\dot{a}_{59} D^2$	$-\dot{a}_{5,10} D$	0	$\dot{a}_{5,12} D$	$-p_{21}$
0	0	0	0	$-\dot{a}_{6,11} D$	0	$-p_{22}$
$\dot{a}_{77} D^2 - a_{77}$	$a_{78}$	$\dot{a}_{79} D$	$\dot{a}_{7,10} D$	0	$-\dot{a}_{7,12} D$	$-p_{23}$
$a_{87}$	$\dot{a}_{88} D^2 - a_{88}$	$-\dot{a}_{89} D$	0	0	$-\dot{a}_{8,12} D$	$-p_{24}$
$-\dot{a}_{97} D$	$\dot{a}_{98} D$	$\dot{a}_{99} D^2$	$\dot{a}_{9,10} D^2$	0	$-\dot{a}_{9,12} D^2$	$-q_{1,1-4}$
$-\dot{a}_{10,7} D$	0	$\dot{a}_{10,9} D^2$	$\dot{a}_{10,10} D^2$	0	$-\dot{a}_{10,12} D^2$	$-q_{2,1-4}$
0	0	0	0	$\dot{a}_{11,11} D^2$	0	$-q_{3,1-4}$
$\dot{a}_{12,7} D$	$\dot{a}_{12,8} D$	$-\dot{a}_{12,9} D^2$	$-\dot{a}_{12,10} D^2$	0	$\dot{a}_{12,12} D^2 - a_{12,12}$	$-q_{4,1-4}$
$a_{68} = a_{88} = \frac{a^2}{2} [\beta_1 h_g - \beta_2 (h - h_g)];$ $\dot{a}_{59} = -\dot{a}_{95} = G [F_2 h_n - F_3 (h - h_n)];$ $\dot{a}_{5,10} = -\dot{a}_{10,5} = G(F_2 + F_3);$ $\dot{a}_{5,12} = -\dot{a}_{12,5} = G [F_2 h_n - F_3 (h - h_n)];$ $a_{63} = 2EF_1 \left( \frac{h^3}{3} - hh_c + h_c^2 \right);$ $a_{86} = 2 [\beta_1 h_c^2 + \beta_2 (h - h_c)^2 + GF_1];$ $\dot{a}_{6,11} = -\dot{a}_{11,6} = 2GF_1;$ $\dot{a}_{77} = \frac{Ea^2}{12} [F_2 h_g^2 + F_3 (h - h_g)^2];$ $a_{77} = \frac{a^2}{2} [\beta_1 h_g^2 + \beta_2 (h - h_g)^2 + G [F_2 h_g^2 + F_3 (h - h_g)^2];$ $a_{78} = a_{87} = \frac{a^2}{2} [\beta_1 h_g^2 + \beta_2 (h - h_g)^2];$ $\dot{a}_{79} = -\dot{a}_{97} = G [F_2 h_n h_g + F_3 (h - h_n) (h - h_g)];$ $\dot{a}_{7,10} = -\dot{a}_{10,7} = G [F_2 h_g - F_3 (h - h_g)];$ $\dot{a}_{7,12} = -\dot{a}_{12,7} = G [F_2 h_g h_n + F_3 (h - h_g) (h - h_n)];$ $\dot{a}_{88} = \frac{EF_1 a^2}{2} \left( \frac{h^3}{3} - hh_g + h_g^2 \right);$ $a_{98} = \frac{a^2}{2} [\beta_1 h_g^2 + \beta_2 (h - h_g)^2 + GF_1];$ $\dot{a}_{89} = -\dot{a}_{98} = GF_1 \frac{a^2}{2};$ $a_{8,12} = -\dot{a}_{12,8} = GF_1 \frac{a^2}{2};$ $\dot{a}_{99} = G \left[ F_2^2 + F_1 \frac{a^2}{2} + F_3 (h - h_n)^2 \right];$ $\dot{a}_{9,10} = \dot{a}_{10,9} = G [F_2 h_n - F_3 (h - h_n)];$ $\dot{a}_{9,12} = \dot{a}_{12,9} = G \left[ F_2 h_n^2 - F_1 \frac{a^2}{2} + F_3 (h - h_n)^2 \right];$ $\dot{a}_{10,10} = G (F_2 + F_3);$ $\dot{a}_{10,12} = \dot{a}_{12,10} = G [F_2 h_n - F_3 (h - h_n)];$ $\dot{a}_{11,11} = 2GF_1;$ $\dot{a}_{12,12} = G \left[ F_2 h_n^2 + F_1 \frac{a^2}{2} + F_3 (h - h_n)^2 \right];$ $a_{12,12} = \frac{a}{3EJ_2 J_3} (J_3 M_1^2 + J_2 M_2^2) + \frac{2h}{3EJ_1} (M_1^2 - M_1 M_2 + M_2^2)$						

A) Number of equations; B) free terms; C) operator.

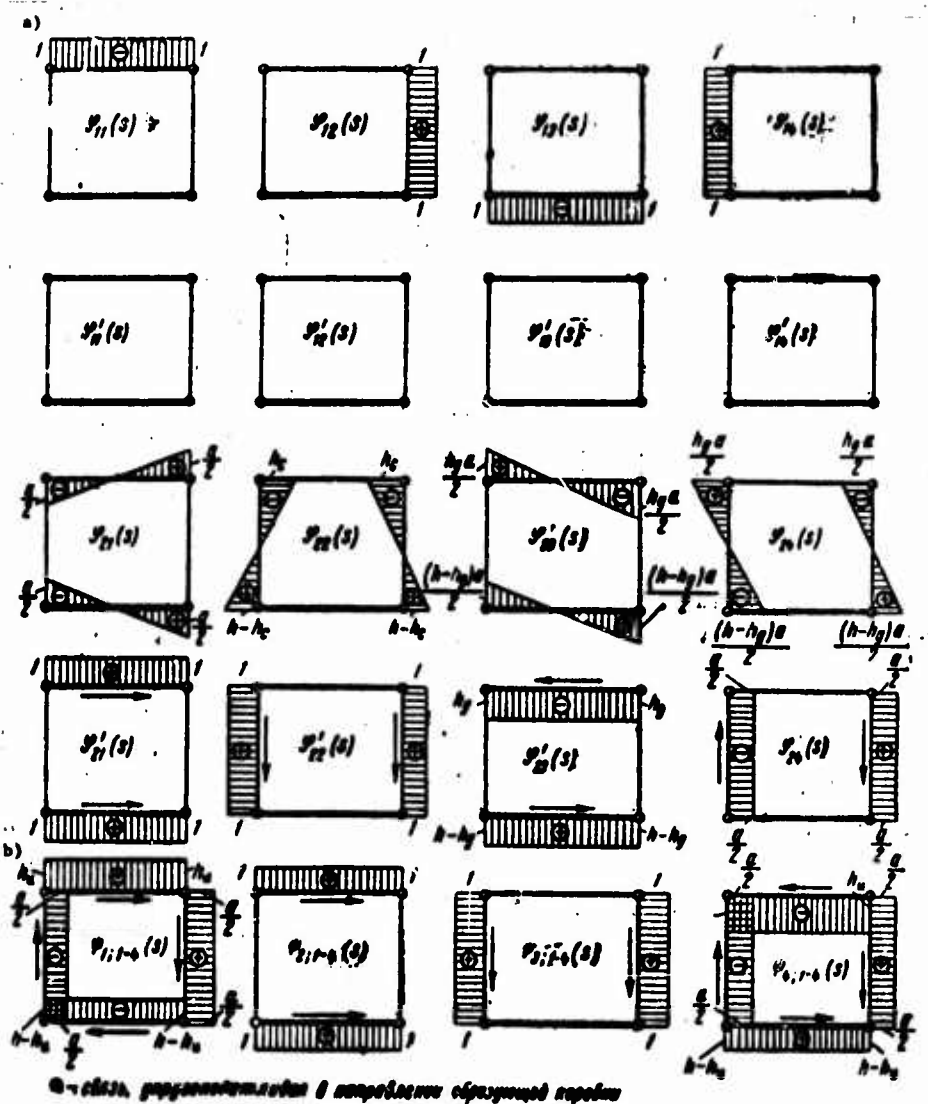


Fig. 2. A) Coupling which is elastocompliant in the direction of the generatrix of the box.

We note that if it will be subsequently necessary (to refine the solution) to assume curvilinear functions of the transverse distribution of displacements, instead of the linear functions, then this will only involve changing the values of the constant coefficients of the sought functions.

The equation obtained above may be simplified by breaking up the complete system of differential equations into individual independent systems. This, in its turn, is achieved by orthogonalizing the functions of the transverse distribution of translations and by breaking up the three-dimensional load into loads which produce only compression (tension), flexure or torsion in the box structure.

The above operation becomes possible when the coordinates of characteristic points of the structure's cross section are known, i.e., of the centers of gravity, flexure and warping. It is obvi-

ous that in a structure whose cross section has one axis of symmetry these characteristic points lie on this axis. If the axis of symmetry and the line of action of the resultant of external forces are parallel, then it becomes possible to break up the external loads into individual components without knowing the second coordinate of the characteristic points and instead of a complex stressed-strained state to examine separately the compression (tension), flexure and torsion states.

## §2. Flexure of Box Structures

### 1. Composite structures

We consider the bending of the structure shown in Fig. 1. To somewhat simplify the problem we assume that the rigidity of couplings resisting longitudinal shear in the upper and lower seams are the same ( $\beta_1 = \beta_2 = \beta$ ).

The revolving system of equations may be obtained from the system shown in Table 1. For this it is necessary to retain in the latter terms located at the intersections of the columns and rows corresponding to those generalized translations which characterize the flexural state of the structure. In our problem these functions are  $U_{11}$ ,  $U_{13}$ ,  $U_{22}$  and  $V_{3,1-4}$ .

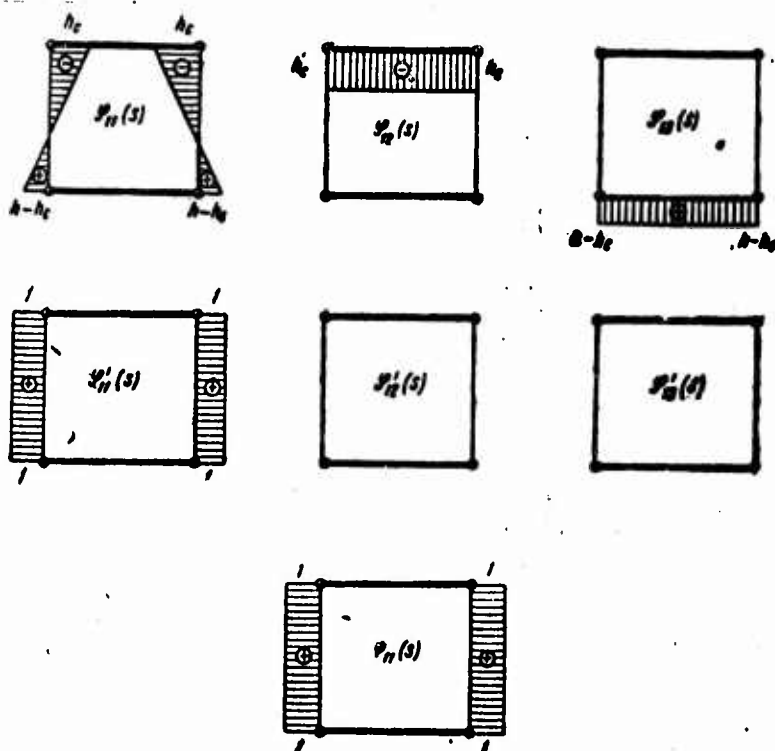


Fig. 3

Selecting the functions of transverse distribution of translations in the manner shown in Fig. 3, and replacing the notations  $U_{22}$  by  $U_{11}$ ,  $U_{11}$  by  $U_{12}$  and  $V_{3,1-4}$  by  $V_{11}$ , we get the resolving system of equations in the form

$$\left. \begin{aligned} a'_{11} U''_{11} - a_{11} U'_{11} + a_{12} U_{12} + a_{13} U_{13} - a'_{14} V'_{11} + p_{11} &= 0; \\ a_{21} U_{11} + a'_{22} U''_{12} - a_{22} U_{12} + p_{12} &= 0; \\ a_{31} U_{11} + a'_{33} U''_{13} - a_{33} U_{13} + p_{13} &= 0; \\ a'_{41} U'_{11} + a'_{44} V''_{11} + q_{11} &= 0, \end{aligned} \right\} (4)$$

where

$$\left. \begin{aligned} a'_{11} &= 2EF_1 \left( \frac{h^2}{3} - hh_c + h_c^2 \right) = 2EJ_{x_1}; & a'_{22} &= EF_1 h_c^2 = EJ_{x_2}; \\ a_{11} &= 2F_1 G + 2\beta [(h-h_c)^2 + h_c^2]; & a_{22} &= 2\beta h_c^2; \\ a_{12} &= a_{21} = 2\beta h_c^2; & a'_{33} &= EF_3 (h-h_c)^2 = EJ_{x_3}; \\ a_{13} &= a_{31} = 2\beta (h-h_c)^2; & a_{33} &= 2\beta (h-h_c)^2; \\ a'_{14} &= a'_{41} = 2F_1 G; & a'_{44} &= 2F_1 G; \end{aligned} \right\} (5)$$

$F_1 = h\delta_1$ ;  $F_2 = a\delta_2$ ;  $F_3 = a\delta_3$  are the cross-sectional areas, respectively, of the top, lateral and bottom plate-facets;  
 $h_c$  is the distance from the neutral axis of the cross section of the box structure to the axis of the top facet.

System (4) is reduced to the single differential equation

$$U''_{12} + a_1 U'_{12} + a_2 U_{12} = K, \quad (6)$$

where

$$K = a_3 q_{11} + a_4 q'_{11} + a_5 p'''_{11} + a_6 p'_{11} + a_7 p^V_{12} + a_8 p'''_{12} + a_9 p'_{12} + a_{10} p_{13};$$

$a_1, a_2, \dots, a_{10}$  are functions of Coefficients (5).

The total solution of the inhomogeneous differential equation (6) is written (with  $K = \text{const}$ ) in the form

$$U_{12} = C_1 \text{sh } r_1 z + C_2 \text{ch } r_1 z + C_3 \text{sh } r_2 z + C_4 \text{ch } r_2 z + C_5 z^2 + C_6 z + C_7 + \frac{K}{6a_2} z^3, \quad (7)$$

where  $r_{1,2} = \frac{1}{2} \sqrt{-2a_1 \pm 2\sqrt{a_1^2 - 4a_2}}$  are the roots of the corresponding characteristic equation, which are real numbers.

For all the systems of equations examined below in the present article (as was shown by numerous solutions of these systems in designing various actual structures), the roots of the corresponding characteristic equations are real.

The remaining unknown functions are determined in terms of the found function  $U_{12}$  using the following expressions

$$\left. \begin{aligned} U_{11} &= \frac{a_{22}}{a_{21}} U_{12} - \frac{a'_{22}}{a_{21}} U'_{12} - \frac{p_{12}}{a_{21}}; \\ U_{13} &= -\frac{a'_{11}}{a_{13}} U'_{11} + \frac{a_{11}}{a_{13}} U_{11} - \frac{a_{12}}{a_{13}} U_{12} + a'_{14} V'_{11} - p_{11}; \\ V_{11} &= -\frac{1}{a_{21} a_{44}} \int \int [a'_{44} (a_{22} U'_{12} - a'_{22} U'''_{12} - p'_{12}) + a_{21} q_{11}] dz + C_8 z + C_9. \end{aligned} \right\} (8)$$

When Expressions (7) and (8) are substituted into System (4), three equations are satisfied identically, while from the fourth equation we obtain an additional relationship which relates linearly certain of the arbitrary constants. The number of arbitrary constants is thus reduced by unity and becomes equal to eight.

The arbitrary constants are determined from boundary conditions, which depend on the character of the end supports of the box structure.

We consider two types of end conditions:

a) stationary end, i.e., the end diaphragm prevents mutual displacement of the plate-facets in the seams;

b) free displacement of the end, i.e., the end diaphragm does not prevent mutual displacement of the plate-facets in the seams.

In both these cases of end supports the end diaphragm is regarded as perfectly rigid in its plane.

The boundary conditions are written on the basis of the following considerations:

in the case of a stationary end (with coordinate  $z=z_1$ ):

a) absence of mutual displacement of the facets on the end results in mutual equality between the generalized longitudinal translations of all the points of end cross sections of the facets of the composite structure

$$U_{1s}(z_1) = U_{12}(z_1) = U_{11}(z_1);$$

b) the end diaphragm may prevent longitudinal mutual displacement of the facets only due to appearance of forces which are normal to the facet cross sections (Fig. 4). But since for a freely supported structure and in the absence of external force effects on the end diaphragm the above system of normal forces must be self-balanced, then we find from the equality to zero of the bending moment in the support cross section

$$\begin{aligned} M(z_1) = \int_F \sigma(z; s) y(s) dF = E \left[ \delta_1 \frac{\partial U_{11}(z_1)}{\partial z} \int_{-h_c}^{h-h_c} y^2 dy + \delta_2 \frac{\partial U_{12}(z_1)}{\partial z} (-h_c)^2 \int_{-\frac{a}{2}}^{+\frac{a}{2}} dx + \right. \\ \left. + \delta_3 \frac{\partial U_{13}(z_1)}{\partial z} (h-h_c)^2 \int_{-\frac{a}{2}}^{+\frac{a}{2}} dx \right] = 2F_1 \left( \frac{h^3}{3} - hh_c + h_c^2 \right) \frac{\partial U_{11}(z_1)}{\partial z} + \\ + F_2 h_c^2 \frac{\partial U_{12}(z_1)}{\partial z} + F_3 (h-h_c)^2 \frac{\partial U_{13}(z_1)}{\partial z} = 0 \end{aligned}$$

or

$$2J_{x_1} \frac{\partial U_{11}(z_1)}{\partial z} + J_{x_2} \frac{\partial U_{12}(z_1)}{\partial z} + J_{x_3} \frac{\partial U_{13}(z_1)}{\partial z} = 0;$$

in the case of free displacement of the end (with coordinate  $z=z_1$ ):

Since the end diaphragm does not interfere with the mutual displacement of the facets, then the normal stresses at the ends of all the facets should be zero. From this

$$\frac{\partial U_{11}(z_1)}{\partial z} = \frac{\partial U_{12}(z_1)}{\partial z} = \frac{\partial U_{13}(z_1)}{\partial z} = 0.$$

The fact that the structure's supports do not move in the vertical direction makes it possible to write the fourth condition for both types of end conditions  $V_{11}(z_1)=0$ .

Thus, for each type of end condition we have four boundary conditions:

in the case of a stationary end

$$\left. \begin{aligned} U_{12}(z_1) &= U_{13}(z_1) = U_{11}(z_1); \\ 2J_{x_1} \frac{\partial U_{11}(z_1)}{\partial z} + J_{x_1} \frac{\partial U_{12}(z_1)}{\partial z} + J_{x_1} \frac{\partial U_{13}(z_1)}{\partial z} &= 0; \quad V_{11}(z_1) = 0; \end{aligned} \right\} \quad (9)$$

in the case of free displacement of ends

$$\frac{\partial U_{11}(z_1)}{\partial z} = \frac{\partial U_{12}(z_1)}{\partial z} = \frac{\partial U_{13}(z_1)}{\partial z} = 0; \quad V_{11}(z_1) = 0. \quad (10)$$

On bending of a composite box structure whose cross section has two axes of symmetry (then  $\delta_2 = \delta_3$ ;  $h_c = h - h_c = \frac{h}{2}$ , such a structure will be henceforth called a symmetrical box structure), unlike a structure whose cross section has one vertical axis of symmetry (asymmetric box structure), the longitudinal translations of the upper and lower facets are of equal magnitude and of opposite sign. It follows from this that  $U_{12} = U_{13}$  and the functions of the transverse distribution of displacements may be selected in the manner shown in Fig. 5.

It is necessary to note that the above selection of function of transverse distribution of displacements somewhat reduces the class of loads for which it is possible to design structures as compared with the class defined by functions selected for analysis of asymmetric structures (Fig. 3). But at the same time this selection makes it possible to obtain a simpler resolving system of equations.

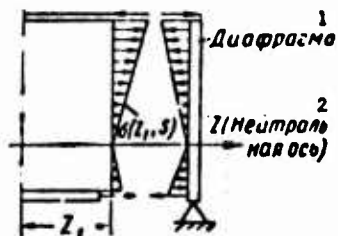


Fig. 4. 1) Diaphragm; 2) neutral axis.

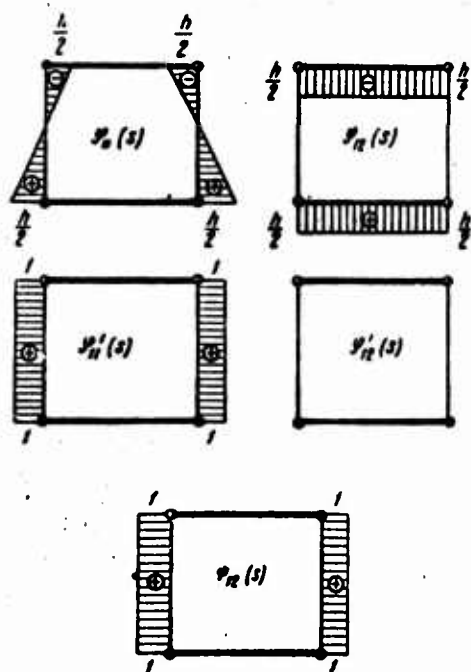


Fig. 5

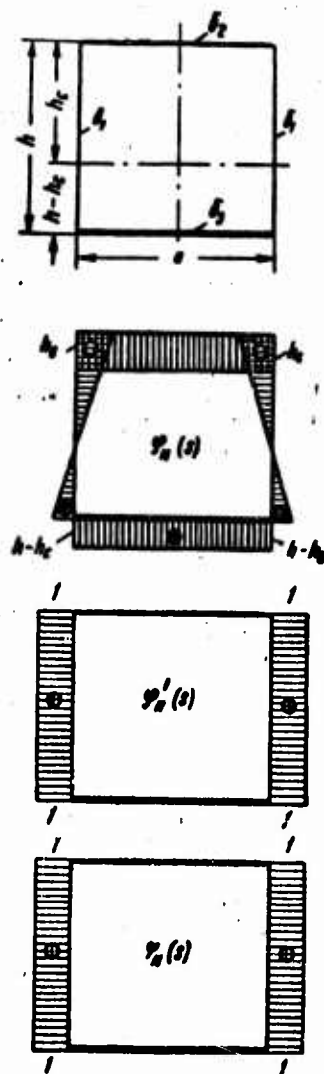


Fig. 6

System (4) for a symmetric box structure is written in the form

$$\left. \begin{aligned} a'_{11} U''_{11} - a_{11} U_{11} + a_{12} U_{12} - a'_{13} V'_{11} + p_{11} &= 0; \\ a_{21} U_{11} + a'_{22} U''_{12} - a_{22} U_{12} + p_{12} &= 0; \\ a'_{31} U'_{11} + a'_{33} V''_{11} + q_{11} &= 0, \end{aligned} \right\} \quad (11)$$

where

$$\begin{aligned} a'_{11} &= EF_1 \frac{h^3}{6} = 2EJ_{x_1}; & a'_{22} &= EF_2 \frac{h^3}{2} = \hat{E}J_{x_1}; \\ a_{11} &= 2F_1 G + \beta h^3; & a_{22} &= \beta h^3; \\ a_{12} &= a_{21} = \beta h^3; & a'_{33} &= 2F_1 G; \\ a'_{13} &= a'_{31} = 2F_1 G; \end{aligned}$$

System (11) reduces to a single differential equation

$$U''_{12} - a_1 U'''_{12} = K, \quad (12)$$

where

$$\left. \begin{aligned} \alpha_1 &= \frac{2(F_1 + 3F_2)}{EF_1 F_2} \beta; \\ K &= \frac{12\beta}{E^2 F_1 F_2 h^3} q_{11} - \frac{2}{h^3 EF_2} \rho_{12}^{III} + \frac{12\beta}{E^2 h^3 F_1 F_2} \rho_{12}^I + \frac{12\beta}{h^3 E^2 F_1 F_2} \rho_{11} \end{aligned} \right\} \quad (13)$$

The total solution of Eq. (12) with  $K = \text{const}$  has the form

$$U_{12} = C_1 \text{sh} \sqrt{\alpha_1} z + C_2 \text{ch} \sqrt{\alpha_1} z + C_3 z^3 + C_4 z + C_5 - \frac{K}{6\alpha_1} z^3. \quad (14)$$

The remaining unknown functions are expressed as follows in terms of the previously found function  $U_{12}$ :

$$\left. \begin{aligned} U_{11} &= -\frac{1}{a_{11}} (a_{22} U_{12}^{II} - a_{22} U_{12} + \rho_{12}); \\ V_{11} &= -\frac{1}{a_{33}} \iint (a_{31} U_{11}^I + q_{11}) dz + C_6 z + C_7. \end{aligned} \right\} \quad (15)$$

As in the case of solving System (4), by substituting Expressions (14) and (15) into one of the equations of starting system (11), we will find the additional relationship between certain of the unknown constants, which is actually the seventh equation (the six equations are obtained from boundary conditions at both ends of the box structure), which is needed for completing the system from which the arbitrary constants are determined.

The boundary conditions for the symmetric structure are written as:

in the case of a stationary end

$$U_{11}(z_1) = U_{12}(z_1); \quad J_{x_1} \frac{\partial U_{11}(z_1)}{\partial z} + J_{x_2} \frac{\partial U_{12}(z_1)}{\partial z} = 0; \quad V_{11}(z_1) = 0; \quad (16)$$

in the case of a freely moving end

$$\frac{\partial U_{11}(z_1)}{\partial z} = \frac{\partial U_{12}(z_1)}{\partial z} = 0; \quad V_{11}(z_1) = 0. \quad (17)$$

## 2. Monolithic structures

The resolving system of equations for an asymmetric monolithic box structure in flexure can be obtained easily from System (4), which defines the stressed-strained state of a composite asymmetric structure in flexure.

Since in the monolithic structure all the couplings between longitudinal facets are perfectly rigid, thus eliminating mutual displacements of facets, then those of Coefficients (5) which contain  $\beta$  as a cofactor vanish, while the sought functions  $U_{11}$ ,  $U_{12}$  and  $U_{13}$  are found to be equal to one another.

Here, if we select functions of the transverse distribution of translations in the manner shown in Fig. 6, then System (4) will be written in the form

$$\left. \begin{aligned} EJ_x U_{11}'' - 2F_1 G (U_{11} + V_{11}') + p_{11} &= 0, \\ 2F_1 G (U_{11}' + V_{11}'') + q_{11} &= 0, \end{aligned} \right\} \quad (18)$$

where

$$EJ_x = 2EJ_{x_1} + EJ_{x_2} + EJ_{x_3} = 2EF_1 \left( \frac{h^3}{3} - hh_c + h_c^2 \right) + EF_2 h_c^2 + EF_3 (h - h_c)^2.$$

In the case of a symmetric structure the resolving system of equations has the same form as System (18). But since for the cross section of a symmetric structure  $\delta_2 = \delta_3$  and  $h_c = \frac{h}{2}$ , then coefficient  $EJ_x$  is written somewhat differently:  $EJ_x = \frac{Eh^3}{6} (F_1 + 3F_2)$ .

For this value of this coefficient System (18) is completely identical with the resolving system of equations for the same structure obtained in [1].

Solution of System (18) is not difficult. To determine the four arbitrary constants which appear on integration use should be made of the following boundary conditions (two at each end):

$$\frac{\partial U_{11}(z_1)}{\partial z} = 0; \quad V_{11}(z_1) = 0.$$

### §3. Torsion of Box Structures

#### 1. Composite structures

We now consider the torsion of a structure produced by torsional loads on simultaneous overhanging of one of the corners. The rigidity of couplings resisting longitudinal shear in the upper and lower seams will be assumed the same as in the case of flexure. The resolving system is obtained by using Table 1. For this we retain in the table only those terms which are located at the intersection of columns and rows pertaining to the generalized translations  $U_{23}$ ,  $U_{24}$ ,  $V_{1,1-4}$  and  $V_{4,1-4}$ . These functions fully characterize the torsional state of the structure.

Selecting functions of the transverse distribution of translations in the manner shown in Fig. 7, and substituting notation  $U_{23}$  by  $U_{11}$ ,  $U_{24}$  by  $U_{12}$ ,  $V_{1,1-4}$  by  $V_{11}$  and  $V_{4,1-4}$  by  $V_{12}$ , we find the system of equations

$$\left. \begin{aligned} a_{11}' U_{11}'' - a_{11} U_{11} &+ a_{13} U_{12} &+ a_{13}' V_{11} &- a_{14}' V_{12} &+ p_{11} &= 0; \\ a_{21} U_{11} &+ a_{22}' U_{12}'' - a_{22} U_{12} &- a_{23}' V_{11} &- a_{24}' V_{12} &+ p_{12} &= 0; \\ -a_{31}' U_{11} &+ a_{32}' U_{12} &+ a_{33}' V_{11}'' &- a_{34}' V_{12}'' &+ q_{11} &= 0; \\ a_{41}' U_{11} &+ a_{42}' U_{12} &- a_{43}' V_{11}'' &+ a_{44}' V_{12}'' - a_{44} V_{12} &+ q_{12} &= 0, \end{aligned} \right\} \quad (19)$$

where

$$\begin{aligned}
a_{11} &= \frac{Ea^3}{12} [F_2 h_g^2 + F_3 (h - h_g)^2]; & a_{23} &= -a_{32} = GF_1 \frac{a^3}{2}; \\
a_{11} &= \frac{\beta a^3}{2} [h_g^2 + (h - h_g)^2] + & a_{24} &= -a_{42} = GF_1 \frac{a^3}{2}; \\
&+ G [F_2 h_g^2 + F_3 (h - h_g)^2]; & & \\
a_{12} = a_{21} &= \frac{\beta a^3}{2} [h_g^2 + (h - h_g)^2]; & a_{33} &= G \left[ F_1 \frac{a^3}{2} + F_2 h_n^2 + \right. \\
& & & \left. + F_3 (h - h_n)^2 \right]; \\
a_{13} &= -a_{31} = G [F_2 h_n h_g + & a_{34} &= G \left[ -F_1 \frac{a^3}{2} + F_2 h_n^2 + \right. \\
&+ F_3 (h - h_n)(h - h_g)]; & & \left. + F_3 (h - h_n)^2 \right]; \\
a_{14} &= -a_{41} = G [F_2 h_n h_g + & a_{44} &= G \left[ F_1 \frac{a^3}{2} + F_2 h_n^2 + \right. \\
&+ F_3 (h - h_n)(h - h_g)]; & & \left. + F_3 (h - h_n)^2 \right]; \\
a_{22} &= \frac{EF_1 a^3}{2} \left( \frac{h^3}{3} - hh_g + h_g^2 \right); & a_{44} &= \frac{a}{3EJ_3 J_2} (J_3 M_1^2 + J_2 M_2^2) + \\
& & & + \frac{2h}{3EJ_1} (M_1^2 - M_1 M_2 + M_2^2). \\
a_{22} &= \frac{\beta a^3}{2} [h_g^2 + (h - h_g)^2] + \frac{a^3}{2} GF_1
\end{aligned} \tag{20}$$

For the case when the transverse diaphragms are rigid in their plane and are placed at a distance from one another such that the contour of the cross section of the structure on torsion may be regarded as undeformed [i.e., function  $\psi_{12}(s)=0$ ], we get a simpler system of equations

$$\left. \begin{aligned}
a_{11} U_{11}^{II} - a_{11} U_{11} &+ a_{12} U_{12} &+ a_{13} V_{11}^I + p_{11} &= 0; \\
a_{21} U_{11} &+ a_{22} U_{12}^{II} - a_{22} U_{12} &- a_{23} V_{11}^I + p_{12} &= 0; \\
-a_{31} U_{11}^I &+ a_{32} U_{12}^I &+ a_{33} V_{11}^{II} + q_{11} &= 0.
\end{aligned} \right\} \tag{21}$$

Using the operator notation for System (19), we reduce it to a single equation

$$U_{11}^{IX} + a_1 U_{11}^{VII} + a_2 U_{11}^V + a_3 U_{11}^{III} + a_4 U_{11}^I = K, \tag{22}$$

where  $a_1, \dots, a_4$  and  $K$  are quite cumbersome expressions.

Solution of Eq. (22) is quite simple in principle, but involves a large volume of computations. Hence for solving various actual problems, it is apparently expedient to use modern computers.

System (21) reduces to the equation

$$U_{11}^V + a_1 U_{11}^{III} + a_2 U_{11}^I = K, \tag{23}$$

where

$$\begin{aligned}
 \alpha_1 &= \frac{1}{a_{11} a_{22} a_{33}} \left[ -a_{11} a_{22} a_{33} + (a'_{13})^2 a_{22} - a'_{11} a_{22} a_{33} + a'_{11} (a'_{23})^2 \right]; \\
 \alpha_2 &= \frac{1}{a_{11} a_{22} a_{33}} \left[ 2a_{21} a'_{32} a'_{13} - (a'_{12})^2 a_{33} + a_{11} a_{22} a_{33} - a_{22} (a'_{13})^2 - (a'_{23})^2 a_{11} \right]; \\
 K &= -\frac{1}{a_{11} a_{22} a_{33}} \left[ a'_{13} a'_{32} p_{11}'' + (a'_{32} a'_{13} - a_{33} a_{11}) p_{11} - a'_{13} a_{22} q + \right. \\
 &\quad \left. + (a_{22} a'_{13} - a_{23} a_{12}) q_{11} \right].
 \end{aligned} \quad (24)$$

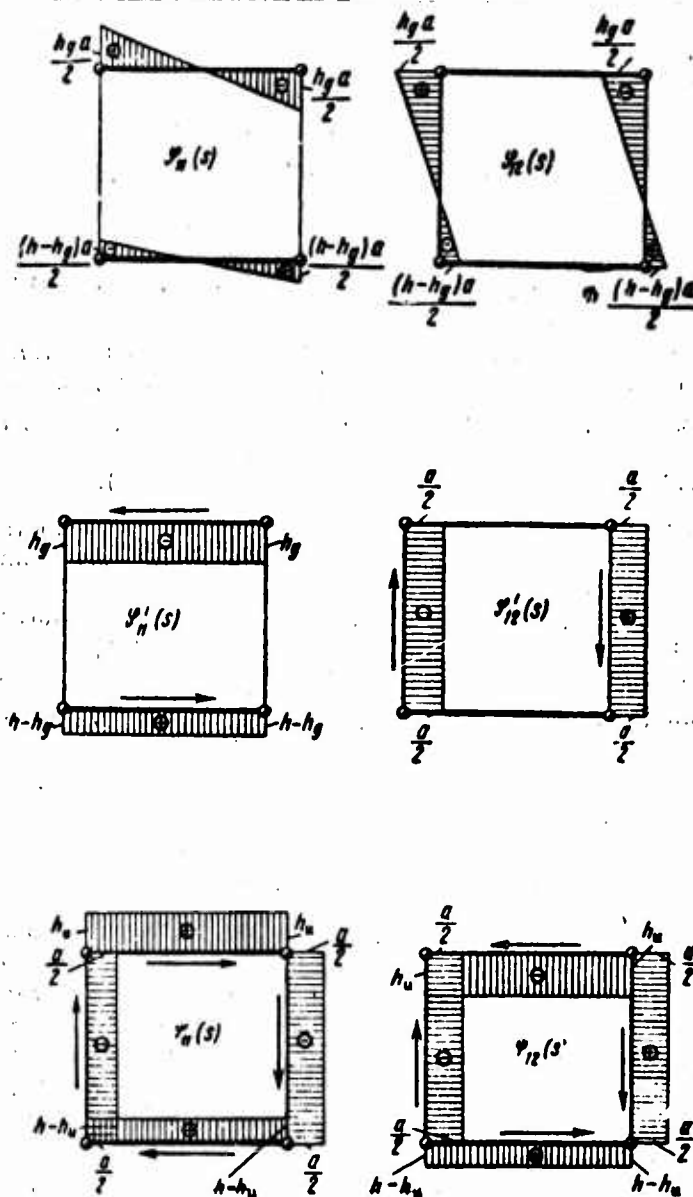


Fig. 7

The total solution of Eq. (23) is written as follows ( $K = \text{const}$ ):

$$U_{11} = C_1 \text{sh } r_1 z + C_2 \text{ch } r_1 z + C_3 \text{sh } r_2 z + C_4 \text{ch } r_2 z + C_5 + \frac{K}{a_1} z, \quad (25)$$

where  $r_{1,2} = \sqrt{-\frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}}$  are the roots of the corresponding characteristic equation.

The remaining unknown functions are defined by the expressions

$$\left. \begin{aligned} U_{12} &= -\frac{1}{a_{12}} (a_{11}'' U_{11}'' - a_{11} U_{11} - a_{13}' V_{11}' + p_{11}); \\ V_{11} &= \iint (b_{11} U_{11}'' + b_2 U_{11}' + b_3 q_{11} + b_4 p_{11}') dz + C_6 z + C_7, \end{aligned} \right\} \quad (26)$$

$$b_1 = -A a_{32}' a_{11}'; \quad b_2 = A (a_{32}' a_{11} - a_{31}' a_{12}); \quad b_3 = A a_{12}; \quad b_4 = -A a_{32}';$$

$$A = \frac{1}{a_{32}' a_{13}' - a_{33}' a_{12}}. \quad (27)$$

Substituting the expressions for generalized translations into the equations of the starting system, we get an additional relationship which is needed for completing the system from which the arbitrary constants are determined.

As in the case of flexure, for the structure under torsion we consider (depending on the character of end supports) two types of boundary conditions.

The boundary conditions are written on the basis of the following considerations:

In the case of a stationary end, the absence of mutual displacement of edges at the end means that the longitudinal translations of all the points of end cross section of the longitudinal plate-facets of the composite structure are zero. The longitudinal displacements which arise due to warping of the construction's cross section may be disregarded in the case at hand, since their magnitude in actual structures, as shown by theoretical and experimental studies, is small as compared with longitudinal translations due to mutual displacement of facets. On the basis of the above we have

$$U_{11}(z_1) = U_{12}(z_1) = 0;$$

In the case of a freely moving end, since the end diaphragm does not prevent mutual displacement of the facets, the normal stresses at the ends should be zero, i.e.,

$$\frac{\partial U_{11}(z_1)}{\partial z} = \frac{\partial U_{12}(z_1)}{\partial z} = 0.$$

If the end diaphragm is rigid in its plane, then irrespective of the manner in which it is fastened to the longitudinal facets, we get the condition

$$V_{12}(z_1) = 0.$$

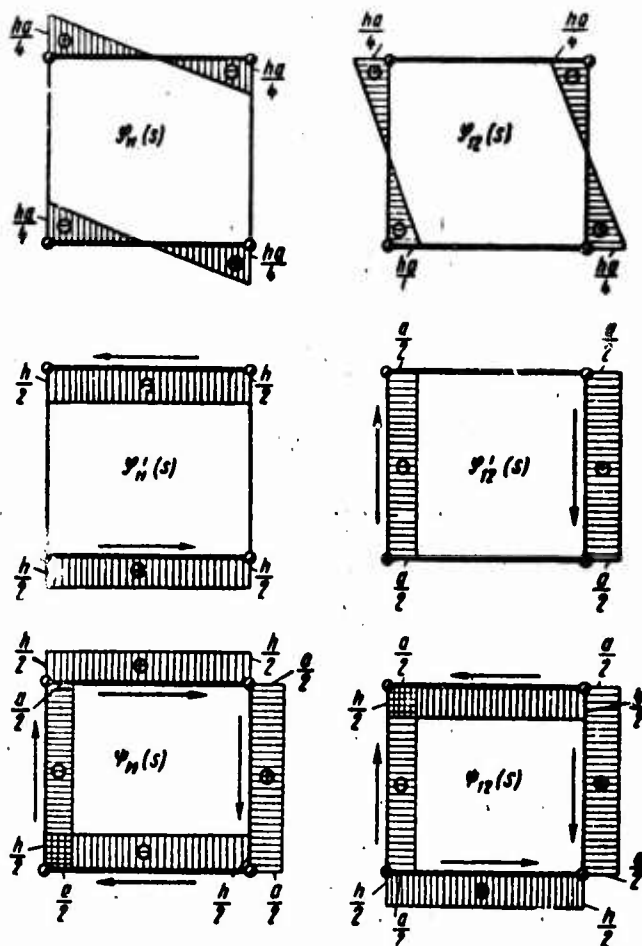


Fig. 8

The existence or lack of existence of rotation of the end cross section on overhanging of one of the supports by an amount  $\Delta$  yields the last condition

$$V_{11}(z_1) = 0 \quad \text{or} \quad V_{11}(z_1) = \frac{\Delta}{a} = \varphi.$$

For each type of end supports we thus have four boundary conditions:

for the case of a stationary end

$$U_{11}(z_1) = U_{12}(z_1) = 0; \quad V_{11}(z_1) = 0 \quad [V_{11}(z_1) = \varphi]; \quad V_{12}(z_1) = 0; \quad (28)$$

for the case of an end free to move

$$\frac{\partial U_{11}(z_1)}{\partial z} = \frac{\partial U_{12}(z_1)}{\partial z} = 0; \quad V_{11}(z_1) = 0 \quad [V_{11}(z_1) = \varphi]; \quad V_{12}(z_1) = 0. \quad (29)$$

The third condition vanishes for structures with an undeformable contour.

The solution of the problem of torsion of symmetrical structure (for which  $\delta_2 = \delta_3$  and  $h_u = h_g = h_c = \frac{h}{2}$ ) does not differ in principle from the torsion problem for an asymmetrical structure. A certain, purely external difference consists in the fact that, due to the possibility which appeared of selecting functions of the transverse distribution of translations in the form shown in Fig. 8, the expressions for the coefficients of unknown functions in Systems (19) and (21) are somewhat simplified.

## 2. Monolithic structures

The resolving system of equations for a monolithic structure is obtained from System (19) by assuming in the latter that the coefficients of the unknown functions which contain as cofactor the quantity  $\beta$  are zero, while the functions of the transverse distribution of translations are selected in the manner shown in Fig. 9.

The system will be written in the following final form

$$\left. \begin{aligned} a'_{11}U''_{11} - a_{11}U_{11} - a'_{12}V''_{11} - a'_{13}V''_{12} + p_{11} &= 0; \\ a'_{21}U''_{11} + a'_{22}V''_{11} - a'_{23}V''_{12} + q_{11} &= 0; \\ a'_{31}U''_{11} - a'_{32}V''_{11} + a'_{33}V''_{12} - a_{33}V_{12} + q_{12} &= 0, \end{aligned} \right\} \quad (30)$$

where

$$\left. \begin{aligned} a'_{11} &= Ea^2 \left[ F_2 \frac{h_g^2}{12} + \frac{F_1}{6} (h^2 - 3hh_g + 3h_g^2) + \frac{F_2}{12} (h - h_g)^2 \right]; \\ a_{11} &= G \left[ F_2 h_g^2 + \frac{F_1 a^2}{6} + F_3 (h - h_g)^2 \right]; \\ a'_{12} = a'_{21} &= G \left[ F_1 \frac{a^2}{2} - F_2 h_u h_g - F_3 (h - h_g)(h - h_u) \right]; \\ a'_{13} = a'_{31} &= G \left[ F_1 \frac{a^2}{2} + F_2 h_u h_g + F_3 (h - h_g)(h - h_u) \right]; \\ a'_{22} &= G \left[ F_1 \frac{a^2}{2} + F_2 h_u^2 + F_3 (h - h_u)^2 \right]; \\ a'_{23} = a'_{32} &= G \left[ F_2 h_u^2 - F_1 \frac{a^2}{2} + F_3 (h - h_u)^2 \right]; \\ a'_{33} &= G \left[ F_2 h_u^2 + F_1 \frac{a^2}{2} + F_3 (h - h_u)^2 \right]; \\ a_{33} &= \frac{a}{3EJ_u J_s} (J_s M_1^2 + J_s M_2^2) + \frac{2h}{3EJ_1} (M_1^2 - M_1 M_2 + M_2^2). \end{aligned} \right\} \quad (31)$$

System (30) is reduced to the single equation

$$U''_{11} + a_1 U'''_{11} + a_2 U''_{11} = K, \quad (32)$$

where  $a_1$ ,  $a_2$  and  $K$  are quite simple functions of Coefficients (31).

Equation (32) is solved without difficulty and its solution is not presented here.

If the cross-sectional contour of the monolithic structure is unvariable, then the resolving system of equations is highly simplified and takes on the form

$$a_{11}' U_{11}'' - a_{11} U_{11} - a_{12}' V_{11}'' + p_{11} = 0;$$

$$a_{21}' U_{11}' + a_{22}' V_{11}'' + q_{11} = 0.$$

The boundary conditions for a monolithic structure with a deformable contour subjected to torsion are written as:

for the case of a diaphragm which is rigid inside as well as outside of its plane

$$U_{11}(z_1) = 0; \quad V_{11}(z_1) = 0 [V_{11}(z_1) = \varphi]; \quad V_{12}(z_1) = 0;$$

for the case of a diaphragm elastic outside and rigid inside the plane

$$\frac{\partial U_{11}(z_1)}{\partial z} = 0; \quad V_{11}(z_1) = 0 [V_{11}(z_1) = \varphi]; \quad V_{12}(z_1) = 0.$$

The resolving system, its solution and the boundary conditions for a symmetrical structure remain in general the same as for an asymmetric structure. Only the value of coefficients of the unknown functions of the system of equations change due to the fact that the graphs of functions of the transverse distribution of translations simplify somewhat.

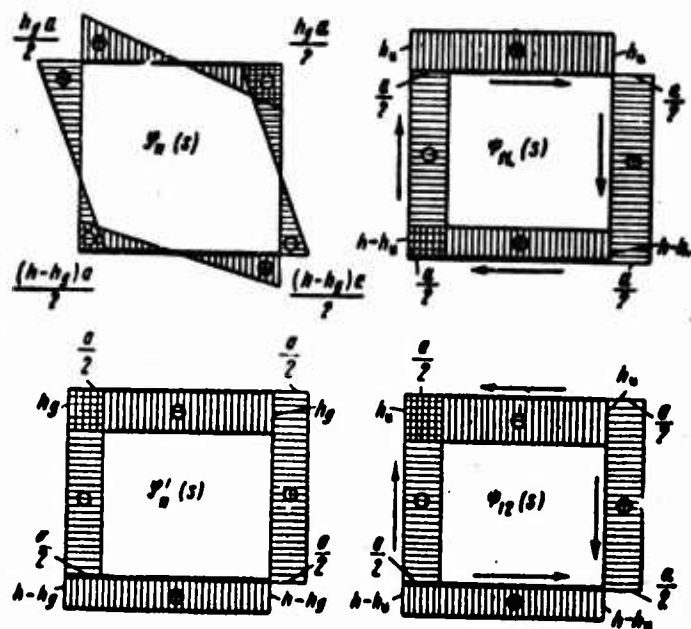


Fig. 9

We note that System (30) is identical with System (15) obtained in [1] for the torsion of a monolithic, thin-walled prismatic shell with a symmetrical rectangular cross section.

#### §4. Concerning the Three Characteristic Points of the Cross Section of Box Structures

As was pointed out toward the end of §1 of the present article, to orthogonalize the functions of transverse distribution of

translations when designing box structures to resist three-dimensional loading, it is necessary to know the location of three characteristic points of the cross section.

In the cross section of monolithic structures these points are the centers of gravity, flexure and warping.

In a composite structure the position of points which may by convention be called the center of gravity, center of flexure and center of warping depends not only on the geometry of the cross section and the physical characteristics of the structure's material (as in a monolithic structure), but also on the physical properties of couplings resisting the longitudinal shear, which interconnect the individual facets.

We shall list the basic properties and attempt to give definitions of the concepts of centers of gravity, flexure and warping in which, in our opinion, are reflected the features peculiar to monolithic as well as composite box structures and which may subsequently serve as a basis of methods for finding the position of characteristic points in the cross sections of composite structures.

The center of gravity is the point of application of the resultant of internal normal forces which arise in the structure's cross section on tension (compression). If the resultants of external forces in all the cross sections are applied in the centers of gravity of the latter, then the structure is subjected only to longitudinal compression or tension without flexure and torsion. The locus of the centers of gravity of the cross section is the neutral axis of the structure for flexure.

The center of flexure is the point of application of the resultant of internal tangential forces, which arise in the structure's cross section on bending. If the resultant of external forces passes through this point, then the bending of the structure is not accompanied by torsion.

The point of intersection of two straight lines, connecting the points of zero normal stresses (which arise in the cross section on constrained torsion) in parallel facets, will be called the center of warping of the cross section of the box structure.

Determination of the positions of centers of gravity and of flexure of cross sections of monolithic structures is not difficult (see [3]); hence we shall dwell only on finding the position of the center of warping of a cross section with one vertical axis of symmetry. It is assumed here that the ensemble of normal stresses in an arbitrary cross section, produced solely due to constrained torsion of the structure (Fig. 10) is a self-balanced system of forces. For this system we must satisfy the three conditions

$$M_x = \int_F \sigma(z; s) y dF = 0; \quad M_y = \int_F \sigma(z; s) x dF = 0; \quad N = \int_F \sigma(z; s) dF = 0.$$

The first and third conditions are satisfied for any  $l_g$ ; hence

we shall use the second condition for solving the problem.

Since in any cross section with coordinate  $z=z_1$

$$\sigma(z_1; s) = EU_{11}'(z_1) \varphi_{11}(s) \text{ and } \varphi_{11}(s) = x(s)y(s),$$

then

$$M_y = EU_{11}'(z_1) \int_F x^2(x)y(s) dF = EU_{11}'(z_1) \frac{a^2}{8} \left[ F, (h-2h_g) - \frac{F_2 h_g}{3} + \frac{F_2 (h-h_g)}{3} \right] = 0,$$

But the factor in front of the brackets is not zero; hence equating to zero the expression in the brackets we find

$$h_g = \frac{3F_1 + F_2}{F + 4F_1} h,$$

where  $F$  is the cross-sectional area of the entire section.

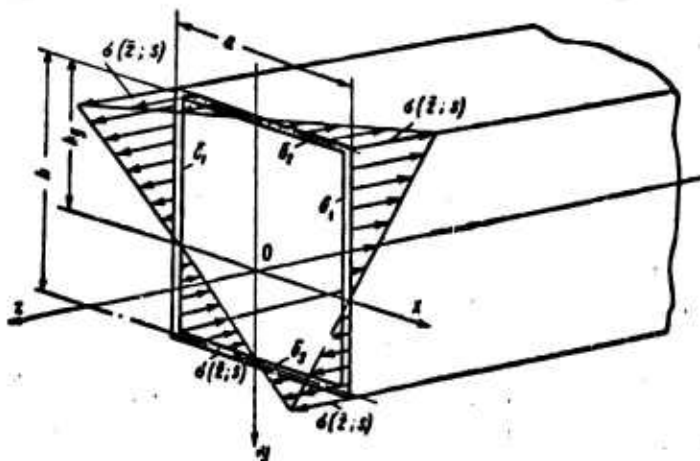


Fig. 10

We shall attempt to determine the position of the center of gravity of the cross section of a composite box structure (Fig. 11) on the assumption that, when the structure is subjected to flexure, the resultants of the internal normal stresses, situated above and below the neutral plane of the cross section are equal to one another, i.e., that the resultant of internal normal forces in any cross section of the structure in flexure is zero

$$\int_F \sigma(z_1; s) = 0.$$

Expanding this condition, we get (the facets are from the same material)

$$E \int_F [U_{11}^I(z_1) \varphi_{11}(s) dF + U_{12}^I(z_1) \varphi_{12}(s) + U_{13}^I(z_1) \varphi_{13}(s)] dF = \\ = Eh [U_{11}^I(z_1) F_1 + U_{13}^I(z_1) F_3] - Eh_c [2U_{11}^I(z_1) F_1 + \\ + U_{12}^I(z_1) F_2 + U_{13}^I(z_1) F_3] = 0,$$

whence

$$h_c = h \frac{U_{11}^I(z_1) F_1 + U_{13}^I(z_1) F_3}{2U_{11}^I(z_1) F_1 + U_{12}^I(z_1) F_2 + U_{13}^I(z_1) F_3}. \quad (33)$$

Since for a monolithic structure  $U_{11}^I(z_1) = U_{12}^I(z_1) = U_{13}^I(z_1)$ , then we find from Formula (33) the expression  $h_c = h \frac{F_1 + F_3}{F}$ , which is identical with the known expression from strength of materials.

As follows from the expressions obtained above, the finding of the coordinates of the centers of gravity, flexure and warping of composite asymmetric structures by the theoretical approach involves certain difficulties and may be the subject of special studies.

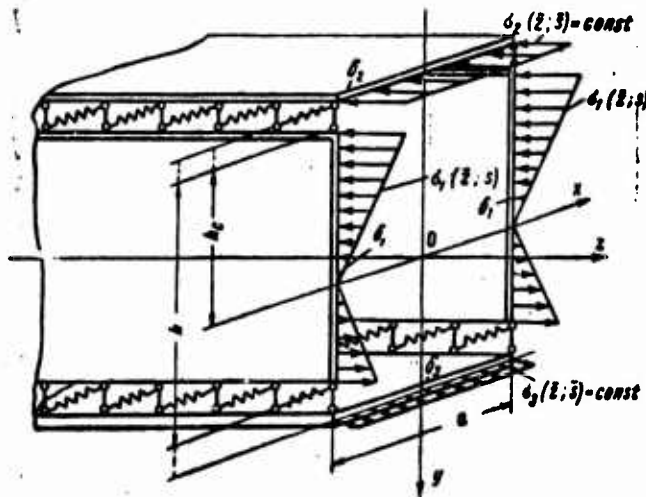


Fig. 11

In symmetric box structures the centers of gravity, flexure and warping are located at the center of symmetry of the cross section, i.e.,  $h_c = h_x = h_g = \frac{h}{2}$ . This situation is valid for monolithic as well as composite structures. Here for composite structures it is also required that the rigidities of the elastocompliant shear-resisting couplings be symmetrical.

## Results

The application of the method for designing composite, thin-walled prismatic structures presented in [2] to the study of composite (knockup) box structures made it possible to find a solution in the general form for a three-dimensional load and solutions in the closed form for certain particular cases of loads and geometry of the structure.

The most important point in solutions using this method is the proper selections of functions of the transverse distribution of translations, which should satisfy the physical meaning of the problem.

The solutions obtained in the present article or those similar to them, may be used (to a certain approximation), for example, for designing composite and monolithic three-dimensional elements of buildings, riveted and welded box beams, composite bridge spans with metal beams and a reinforced-concrete plate, combined with the cross section by means of elastic splines, monocoque aircraft structures with cutouts, bodies of all-metal railroad cars, etc.

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## DESIGN OF BOX-BEAM STRUCTURAL SPANS BY THE METHOD OF TRANSLATIONS

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References [1] and [2] consider the application of the method of translations to the design of plate-beam systems and of wrinkled shells. This method makes it practically possible (within the framework of assumptions of the theory of elasticity) to obtain an exact solution, and it is convenient for programming of electronic digital computers [EDC] (ЭЦМ). Thus, [2] describes a program for designing diaphragmless, multirib beam spans using the "Ural-2" computer, composed by the present author. This program was successfully used in calculations performed in the State Scientific Research Institute on Design of Industrial Transportation Structures and Devices and State Design and Research Institute on Research and Design of Large Bridges of the State Production Committee on Transportation Construction of the USSR. A similar program is being at present composed for designing box-beam spans. Below are presented the principal formulas for designing these systems using the method of translations. The general methodology of obtaining the functions, expressions of special functions used in the calculations, and all the basic notation are not presented here, since they are described in detail in [2].

We now consider a diaphragmless, hinged, box-type structural span, shown in Fig. 1a. This system will be regarded as a combination of plates and girders. The latter are situated at the point of intersection of the plates and may be formed by brackets. The design scheme of the structural span is shown in Fig. 1b. Figure 1c shows the unknown methods of translations, used in calculations. It is assumed that the vertical translations of points of two junction lines (i.e., lines of intersection of the middle planes of the plates) belonging to the same rib are assumed to be equal. Quantities  $Z_1, Z_2, Z_3$ , etc., corresponding to the linear translations of the junction-line points, are here used to denote the angular measurement of these translations, i.e., the ratio of the actual linear displacement to some dimension  $b$ , which is taken as basic.

The coefficient matrix of the system of equations of the method of displacements for a structural span with  $k$  ribs has the following structure

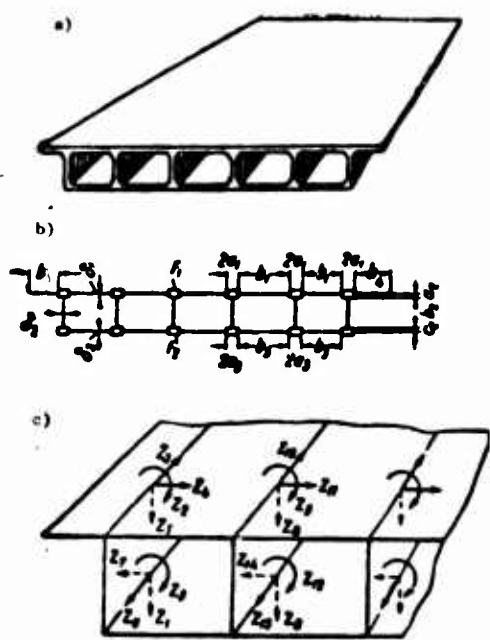


Fig. 1

$$R = \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & R_3 & \\ & & & R_4 \\ & & & & \ddots \\ & & & & & R_k \end{pmatrix}$$

Due to the fact that this system is regular, matrices  $R_2, R_3, \dots, R_{k-1}$  will be identical. These matrices have 7 rows and 21 columns each. Corner matrices  $R_1$  and  $R_k$  have 7 rows and 14 columns each and differ only by the signs of the individual elements, namely elements  $r_{12}^{(n)} = r_{21}^{(n)}, r_{15}^{(n)} = r_{61}^{(n)}, r_{34}^{(n)} = r_{43}^{(n)}, r_{67}^{(n)} = r_{76}^{(n)}$  of matrix  $R_1$  and their corresponding elements of matrix  $R_k$  are of

opposite sign. Below are presented formulas for calculating the coefficients of matrices  $R_1$  and  $R_2$

which, on the basis of the above, makes it possible to write the entire matrix  $R$ . The coefficients are presented in sequence for the rows of matrices  $R_1$  and  $R_2$ , starting in each row from the principal element. The elements located below the principal diagonal are written from symmetry of matrix  $R$

$$\begin{aligned} r_{11}^{(n)} &= A_1 + A_1' + A_1'' = \frac{E\delta_2 b^3}{b_2} \left[ 2\psi_1(\alpha_2) - 2\psi_2(\alpha_2) + \right. \\ &+ \frac{2n\pi(\alpha_2 + c_2)}{l} \left[ \psi_3(\alpha_2) + \psi_4(\alpha_2) \right] + \frac{n^2\pi^2(a_2^2 + c_2^2)}{l^2} \psi_5(\alpha_2); \\ &\left. + \frac{2n^2\pi^2 a_2 c_2}{l^2} \psi_6(\alpha_2) \right] + \sum_{i=1,3} \frac{D_i b^3}{b_i^3} f_5(\alpha_i) + \frac{D_i b^3}{b_i^3} f_6(\alpha_i); \\ r_{12}^{(n)} &= \frac{D_1 b}{b_1^2} \left[ f_2(\alpha_1) + \frac{a_1}{b_1} f_3(\alpha_1) \right]; \\ r_{13}^{(n)} &= -\frac{E\delta_2 b^3}{b_2} \left[ \psi_3(\alpha_2) + \psi_4(\alpha_2) + \frac{n\pi a_2}{l} \psi_5(\alpha_2) + \frac{n\pi c_2}{l} \psi_6(\alpha_2) \right]; \\ r_{14}^{(n)} &= 0; \quad r_{15}^{(n)} = \frac{D_3 b}{b_3^2} \left[ f_2(\alpha_3) + \frac{a_3}{b_3} f_3(\alpha_3) \right]; \\ r_{16}^{(n)} &= -\frac{E\delta_2 b^3}{b_2} \left[ \psi_3(\alpha_2) + \psi_4(\alpha_2) + \frac{n\pi c_2}{l} \psi_5(\alpha_2) + \frac{n\pi a_2}{l} \psi_6(\alpha_2) \right]; \\ r_{17}^{(n)} &= 0; \quad r_{18}^{(n)} = -b^2 \left[ \frac{D_1}{b_1^3} f_6(\alpha_1) + \frac{D_3}{b_3^3} f_6(\alpha_3) \right]; \\ r_{19}^{(n)} &= \frac{D_1 b}{b_1^2} \left[ f_4(\alpha_1) + \frac{a_1}{b_1} f_5(\alpha_1) \right]; \quad r_{1,10}^{(n)} = r_{1,11}^{(n)} = 0; \end{aligned}$$

$$r_{1,12}^{(n)} = \frac{D_3 b}{b_3^2} \left[ f_4(a_2) + \frac{a_2}{b_2} f_6(a_2) \right]; \quad r_{1,13}^{(n)} = r_{1,14}^{(n)} = \dots = 0.$$

$$r_{22}^{(n)} = \frac{n^2 \pi^2 G J_{d1}}{I^2} + \sum_{i=1}^{i=2} \frac{D_i}{b_i} \left[ f_1(a_i) + 2 \frac{a_i}{b_i} f_3(a_i) + \frac{a_i^2}{b_i^2} f_5(a_i) \right] +$$

$$+ \frac{D_1}{b_4} \left[ f_7(a_4) + 2 \frac{a_1}{b_4} f_9(a_4) + \frac{a_1^2}{b_4^2} f_9(a_4) \right];$$

$$r_{23}^{(n)} = 0; \quad r_{24}^{(n)} = -\frac{D_3 b}{b_2^2} \left[ f_3(a_2) + \frac{a_2}{b_2} f_5(a_2) \right];$$

$$r_{25}^{(n)} = \frac{D_2}{b_2} \left[ f_3(a_2) + \frac{a_2 + c_2}{b_2} f_4(a_2) + \frac{a_2 c_2}{b_2^2} f_6(a_2) \right];$$

$$r_{26}^{(n)} = 0; \quad r_{27}^{(n)} = -\frac{D_3 b}{b_2^2} \left[ f_4(a_2) + \frac{a_2}{b_2} f_6(a_2) \right]; \quad r_{28}^{(n)} = -r_{19}^{(n)};$$

$$r_{29}^{(n)} = \frac{D_1}{b_1} \left[ f_2(a_1) + 2 \frac{a_1}{b_1} f_4(a_1) + \frac{a_1^2}{b_1^2} f_6(a_1) \right]; \quad r_{2,10}^{(n)} = r_{2,11}^{(n)} = \dots = 0.$$

$$r_{33}^{(n)} = \frac{n^2 \pi^2 E F_1}{I^2} b^2 + \frac{E \delta_1 b^2}{b_1} \psi_3(a_1) + \frac{E \delta_1 b^2}{b_4} \psi_3(a_4) + \frac{E \delta_2 b^2}{b_2} \psi_3(a_2);$$

$$r_{34}^{(n)} = \frac{E \delta_1 b^2}{b_4} \left[ \psi_3(a_4) + \frac{n \pi a_1}{l} \psi_3(a_4) \right] - \frac{E \delta_1 b^2}{b_1} \left[ \psi_3(a_1) + \frac{n \pi a_1}{l} \psi_3(a_1) \right];$$

$$r_{35}^{(n)} = 0; \quad r_{36}^{(n)} = \frac{E \delta_2 b^2}{b_2} \psi_3(a_2); \quad r_{37}^{(n)} = r_{38}^{(n)} = r_{39}^{(n)} = 0;$$

$$r_{3,10}^{(n)} = -\frac{E \delta_1 b^2}{b_1} \psi_3(a_1);$$

$$r_{3,11}^{(n)} = -\frac{E \delta_1 b^2}{b_1} \left[ \psi_4(a_1) + \frac{n \pi a_1}{l} \psi_4(a_1) \right]; \quad r_{3,12}^{(n)} = \dots = 0.$$

$$r_{44}^{(n)} = A_4 + A_4' + A_4'' = \frac{D_2 b^2}{b_2^2} f_5(a_2) + \frac{E \delta_1 b^2}{b_1} \left[ \psi_1(a_1) + \frac{2 n \pi a_1}{l} \psi_3(a_1) + \right.$$

$$\left. + \frac{n^2 \pi^2 a_1^2}{I^2} \psi_3(a_1) \right] + \frac{E \delta_1 b^2}{b_4} \left[ \psi_7(a_4) + \frac{2 n \pi a_1}{l} \psi_3(a_4) + \frac{n^2 \pi^2 a_1}{I^2} \psi_3(a_4) \right];$$

$$r_{45}^{(n)} = -\frac{D_3 b}{b_2^2} \left[ f_4(a_2) + \frac{c_2}{b_2} f_6(a_2) \right]; \quad r_{46}^{(n)} = 0; \quad r_{47}^{(n)} = \frac{D_2 b^2}{b_2^2} f_6(a_2);$$

$$r_{48}^{(n)} = r_{49}^{(n)} = 0; \quad r_{4,10}^{(n)} = -r_{3,11}^{(n)};$$

$$r_{4,11}^{(n)} = \frac{E \delta_1 b^2}{b_1} \left[ -\psi_2(a_1) + \frac{2 n \pi a_1}{l} \psi_4(a_1) + \frac{n^2 \pi^2 a_1^2}{I^2} \psi_4(a_1) \right];$$

$$r_{4,12}^{(n)} = r_{4,13}^{(n)} = \dots = 0.$$

$$r_{55}^{(n)} = A_5 + A_5' + A_5'' = \frac{n^2 \pi^2 G J_{d1}}{I^2} + \frac{D_3}{b_3} \left[ f_1(a_2) + 2 \frac{c_2}{b_3} f_3(a_2) + \frac{c_2^2}{b_3^2} f_5(a_2) \right] +$$

$$\begin{aligned}
& + \frac{D_3}{b_3} \left[ f_1(a_3) + 2 \frac{a_3}{b_3} f_2(a_3) + \frac{a_3^2}{b_3^2} f_3(a_3) \right]; \quad r_{66}^{(n)} = 0; \\
r_{57}^{(n)} &= -\frac{D_3 b}{b_3^2} \left[ f_3(a_2) + \frac{c_3}{b_2} f_5(a_2) \right]; \quad r_{58}^{(n)} = -r_{1,12}^{(n)}; \quad r_{69}^{(n)} = r_{6,10}^{(n)} = r_{6,11}^{(n)} = 0; \\
r_{5,12}^{(n)} &= \frac{D_3}{b_3} \left[ f_2(a_3) + 2 \frac{a_3}{b_3} f_4(a_3) + \frac{a_3^2}{b_3^2} f_6(a_3) \right]; \quad r_{6,13}^{(n)} = \dots = 0. \\
r_{66}^{(n)} &= A_6 + A_6' + A_6'' = \frac{n^2 \pi^2 E F_1}{l^3} b^2 + \frac{E \delta_3 b^3}{b_3} \psi_3(a_2) + \frac{E \delta_3 b^3}{b_3} \psi_3(a_1); \\
r_{67}^{(n)} &= \frac{E \delta_3 b^3}{b_3} \left[ \psi_3(a_2) + \frac{n \pi a_2}{l} \psi_3(a_3) \right]; \quad r_{68}^{(n)} = r_{69}^{(n)} = \dots = r_{6,12}^{(n)} = 0; \\
r_{6,13}^{(n)} &= -\frac{E \delta_3 b^3}{b_3} \psi_6(a_3); \quad r_{6,14}^{(n)} = -\frac{E \delta_3 b^3}{b_3} \left[ \psi_4(a_2) + \frac{n \pi a_2}{l} \psi_6(a_3) \right]; \quad r_{6,15}^{(n)} = \dots = 0. \\
r_{77}^{(n)} &= A_7 + A_7' + \frac{D_3 b^3}{b_2^3} f_5(a_2) + \frac{E \delta_3 b^3}{b_3} \left[ \psi_1(a_2) + \right. \\
& \quad \left. + \frac{2 n \pi a_2}{l} \psi_3(a_3) + \frac{n^2 \pi^2 a_2^2}{l^2} \psi_5(a_2) \right]; \\
r_{78}^{(n)} &= \dots = r_{7,12}^{(n)} = 0; \quad r_{7,13}^{(n)} = -r_{6,14}^{(n)}; \\
r_{7,14}^{(n)} &= \frac{E \delta_3 b^3}{b_3} \left[ -\psi_2(a_2) + \frac{2 n \pi a_2}{l} \psi_4(a_3) + \frac{n^2 \pi^2 a_2^2}{l^2} \psi_6(a_3) \right]; \quad r_{7,15}^{(n)} = \dots = 0. \\
r_{88}^{(n)} &= A_8 + 2 A_8' (\text{см. } r_{11}^{(n)}); \quad r_{89}^{(n)} = 0; \quad r_{8,10}^{(n)} = r_{13}^{(n)}; \\
r_{8,11}^{(n)} &= r_{8,12}^{(n)} = 0; \quad r_{8,13}^{(n)} = r_{16}^{(n)}; \quad r_{8,14}^{(n)} = 0, \text{ etc., as in row } Z_1^{(n)}. \\
r_{99}^{(n)} &= \frac{n^2 \pi^2 G J_{d1}}{l^3} + 2 \frac{D_1}{b_1} \left[ f_1(a_1) + 2 \frac{a_1}{b_1} f_3(a_1) + \frac{a_1^2}{b_1^2} f_5(a_1) \right] + \\
& + \frac{D_2}{b_2} \left[ f_1(a_2) + 2 \frac{a_2}{b_2} f_3(a_2) + \frac{a_2^2}{b_2^2} f_5(a_2) \right], \text{ etc., as in row } Z_2^{(n)}. \\
r_{10,10}^{(n)} &= \frac{n^2 \pi^2 E F_1}{l^3} b^2 + 2 \frac{E \delta_1 b^3}{b_1} \psi_6(a_1) + \frac{E \delta_2 b^3}{b_2} \psi_6(a_2). \\
r_{10,11}^{(n)} &= 0, \text{ etc., as in row } Z_3^{(n)}. \\
r_{11,11}^{(n)} &= A_4 + 2 A_4' (\text{см. } r_{44}^{(n)}), \text{ etc., as in row } Z_4^{(n)}. \\
r_{12,12}^{(n)} &= A_5 + A_5' + 2 A_5'' (\text{см. } r_{55}^{(n)}), \text{ etc., as in row } Z_5^{(n)}. \\
r_{13,13}^{(n)} &= A_6 + A_6' + 2 A_6'' (\text{см. } r_{66}^{(n)}), \quad r_{13,14}^{(n)} = 0, \text{ etc., as in row } Z_6^{(n)}. \\
r_{14,14}^{(n)} &= A_7 + 2 A_7' (\text{см. } r_{77}^{(n)}), \text{ etc., as in row } Z_7^{(n)}.
\end{aligned}$$

In the above formulas

$$a_i = \frac{n \pi b_i}{2l}; \quad D_i = \frac{E \delta_i^3}{12(1-\nu^2)};$$

where  $n$  is the number of the sine or cosine harmonic under consideration, which specify the translations of the system's points;  $l$  is the span length;  $\nu$  is Poisson's ratio;  $E F_i$ ,  $G J_{di}$  are the tension-compression and torsional rigidities of the junction girders ( $i=1$  for the upper and  $i=2$  for the lower junctions). The balance of the notation is clear from Fig. 1. Functions  $f_i(a)$  and  $\psi_i(a)$  and their tabulated values are presented in [2].

If the system under consideration has diaphragms along its length, then one should use the mixed method of calculations, removing the diaphragms and replacing their effect on the remaining diaphragmless system by corresponding forces  $X_i$ . It is also possible to use the method of translations in its pure form. But then it is necessary in calculating the reactions also to take into account the concentrated reaction forces produced by deformation of the diaphragms. Here all the sine and cosine harmonics which specify the deformations of the system cannot in general be separated, which increases the order of the system of equations. Both above solution approaches are considered in [2].

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## CONCERNING THE DESIGN OF LATTICED-BAR FLOORING SYSTEMS, SUPPORTED ON A RECTANGULAR CONTOUR

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### §1. Basic Considerations

We consider latticed-bar flooring systems (skeletons), supported on a rectangular contour, composed from a sufficiently large number of identical elements, located at equal distances from one another in any direction. Design of such systems by general methods of structural mechanics is a quite complex problem from the computational point of view. However, its solution may be substantially simplified by considering the system under study not as being composed of individual discrete elements, but as some continuum.

The order of the errors in this approach to this type of problem, as was shown by P.F. Papkovich [3] is equal to  $\frac{1}{i^2}$  when finding the magnitudes of moments and to  $\frac{1}{i}$  when determining the system's deflections, where  $i$  is the number of intervals into which the given segment is broken up. Thus, as early as for 5 intervals the accuracy of the solution obtained by replacing the lattice formed by discrete elements by some continuum will fully satisfy the requirements put to the accuracy of engineering calculations.

At the same time, the simplification thus achieved is quite appreciable and makes it possible, as will be shown below, to obtain an approximate solution of a number of complex problems in the closed form.

It should be noted that also other systems of latticed-structure design, using the method of forces or the method of calculating finite differences, may also contain errors of approximately the same orders as given above. The source of these errors here will be the replacement of the actual load distribution among the lattice elements by a continuous uniform load or by a system of concentrated forces.

Let us consider some systems of bar-lattice floorings encountered in practice.

## §2. A Beam and Cable Framework

In the beam and cable framework (Fig. 1) the carrying elements of one direction are freely supported beams, while those of the second direction are support-fastened cables. The outline of latter corresponds to the pressure curve due to the load acting on the flooring, and in the case of load uniformly distributed along the cable the corresponding continuous system which is replacing the beam and cable framework under study has the outlines of a concave parabolic cylinder. We shall consider sufficiently shallow floorings (with a sag  $f$  not exceeding  $1/8$  of the cable span  $s$ ), disregarding the difference between the length of the span and the length of the cable. Henceforth the beam spans are denoted by  $l$  and we consider loads on the system which do not depend on the  $y$  coordinate, i.e., which are constant along the length of the cable.

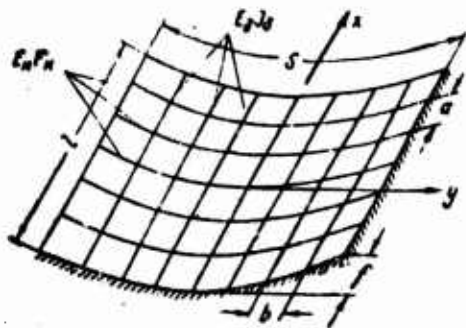


Fig. 1

The tensile stiffnesses of the cables and the flexural rigidities of the beams per unit length, corresponding to the stiffnesses of the replacing continuum, are

$$K = \frac{E_k F_k}{a}, \quad B = \frac{E_b J_b}{b}.$$

The magnitude of the constant horizontal cable tension  $H(x)$  and the continuous uniform load  $q_k(x)$  balancing it are interrelated by the relationship

$$H(x) = \frac{q_k(x) s^2}{8f}. \quad (1)$$

Let us consider a segment of the cable of length  $\Delta s$ . The work of external forces  $q_k(x)$  on cable translations  $w$  for a segment length  $\Delta s$  is

$$\Delta A = q_k(x) w \Delta s. \quad (2)$$

At the same time the work of internal forces  $H(x)$  is defined as

$$\Delta R = - \frac{[H(x)]^2}{K} \Delta s = - \frac{[q_k(x)]^2 s^4}{64f^2 K} \Delta s. \quad (3)$$

Since  $\Delta A = -\Delta R$ , then introducing the notation

$$C = \frac{64f^2 K}{s^4}, \quad (4)$$

we find

$$q_k(x) = Cw. \quad (5)$$

At the same time, the deflection of the framework's beams is related to the loads taken up by the relationship

$$B \frac{\partial^4 w}{\partial x^4} = q_0(x), \quad (6)$$

The total loads  $q_0(x)$  taken up by the beams and  $q_u(x)$  taken up by the cable must be equal to the load  $q(x)$  applied to the lattice. Taking this into account, we arrive at the differential equation of equilibrium

$$B \frac{\partial^4 w}{\partial x^4} + Cw = q(x), \quad (7)$$

which is identical with the differential equation of the deflection of a beam supported on a nonbinding elastic foundation.

We consider the case of a continuous uniform load  $q(x) = p$ . We introduce the notation

$$k = \frac{2}{5} \sqrt[4]{\frac{p^3 K}{B}}, \quad u = \frac{kl}{2} = \frac{l}{2} \sqrt[4]{\frac{p^3 K}{B}}. \quad (8)$$

The solution of Eq. (7) for a freely supported beam subjected to a continuous uniform load may, according to I.G. Bubnov [1], be represented in the form

$$w(x) = \frac{p}{c} [1 - \varphi(u, x)] = \frac{ps^4}{64f^3 K} [1 - \varphi(u, x)], \quad (9)$$

where use is made of the transcendental function

$$\varphi(u, x) = \frac{2(\operatorname{sh} u \sin u \operatorname{sh} kx \sin kx + \operatorname{ch} u \cos u \operatorname{ch} kx \cos kx)}{\operatorname{ch} 2u + \cos 2u}. \quad (10)$$

The framework has its maximum deflection along axis  $x=0$ , where it is

$$w_{\max} = \frac{ps^4}{64f^3 K} \left( 1 - \frac{2\operatorname{ch} u \cos u}{\operatorname{ch} 2u + \cos 2u} \right). \quad (11)$$

Correspondingly, according to (1) and (5), the maximum horizontal tension in the framework's cables per unit length is

$$H_{\max} = \frac{ps^2}{8f} \left( 1 - \frac{2\operatorname{ch} u \cos u}{\operatorname{ch} 2u + \cos 2u} \right). \quad (12)$$

The moments per unit length in the beam are found from the formula

$$M(x) = -Bw''(x) = \frac{Bk^3 p}{C} \varphi''(u, x) = \frac{ps^3}{16} \sqrt{\frac{B}{f^3 K}} \times \\ \times \frac{4(\operatorname{sh} u \sin u \operatorname{ch} kx \cos kx - \operatorname{ch} u \cos u \operatorname{sh} kx \sin kx)}{\operatorname{ch} 2u + \cos 2u}, \quad (13)$$

here they may be at maximum also not in the beam midspans.

Here the solution of the problem resulted in closed formulas. For a load which is not uniform, when we do not have a readymade solution of the problem of deflection of a freely supported beam

lying on a nonbinding elastic foundation, it is possible to find a solution in terms of trigonometric series. Let there exist a sine series expansion

$$q = \sum_{m=1}^{m=\infty} q_m \sin m\pi \left( \frac{x}{l} + 0,5 \right); \quad (14)$$

we shall seek the expression for the framework's deflection also as a sine series

$$w = \sum_{m=1}^{m=\infty} w_m \sin m\pi \left( \frac{x}{l} + 0,5 \right). \quad (15)$$

Substituting these expansions into Eq. (7), we find

$$w_m = \frac{q_m}{C \left( 1 + \frac{m^4 \pi^4 B}{Cl^4} \right)} = \frac{q_m s^4}{64 f^3 K \left( 1 + \frac{m^4 \pi^4 s^4 B}{64 f^3 l^4 K} \right)}. \quad (16)$$

From this we get expansions for the deflection of the lattice, horizontal tensions per unit length in the cables and moments per unit length in the beams

$$\left. \begin{aligned} w &= \sum_{m=1}^{m=\infty} \frac{q_m s^4 \sin m\pi \left( \frac{x}{l} + 0,5 \right)}{64 f^3 K \left( 1 + \frac{m^4 \pi^4 s^4 B}{64 f^3 l^4 K} \right)}; \\ H_k &= \sum_{m=1}^{m=\infty} \frac{q_m s^2 \sin m\pi \left( \frac{x}{l} + 0,5 \right)}{8 f \left( 1 + \frac{m^4 \pi^4 s^4 B}{64 f^3 l^4 K} \right)}; \\ M_\delta &= \sum_{m=1}^{m=\infty} \frac{q_m m^2 \pi^2 s^4 B \sin m\pi \left( \frac{x}{l} + 0,5 \right)}{64 f^3 l^3 K \left( 1 + \frac{m^4 \pi^4 s^4 B}{64 f^3 l^4 K} \right)}. \end{aligned} \right\} \quad (17)$$

Despite the not-too-rapid, particularly for the last expansion, convergence of these series, they are frequently more convenient to use than solving Eq. (7) by standard methods.

### §3. A Lattice of Intersecting Beams (Fig. 2)

Let us consider a lattice of intersecting beams with constant rigidities per unit length in each direction, equal to

$$B_x = \frac{E_x J_x}{l_y}, \quad B_y = \frac{E_y J_y}{l_x}.$$

The loads taken up by each of the system's beams can be found from formulas such as (6). Since the total loads  $q(x)$  and  $q(y)$  in all points of the lattice must be equal to the external load, the differential equation of equilibrium for this system has the form

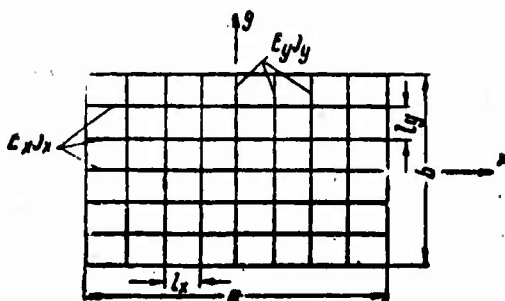


Fig. 2

$$B_x \frac{\partial^4 w}{\partial x^4} + B_y \frac{\partial^4 w}{\partial y^4} = q(x, y). \quad (18)$$

It is assumed here that the beam reactions in the lattice junctions reduces only to vertical forces, which is equivalent to disregarding the beam's torsional stiffness.

Differential equation (18) is most conveniently solved in a series. For the case when the lattice under study is subjected to a continuous uniform load  $q$  it is expedient to use the method of simple series. We introduce the notation

$$k_n = \frac{n\pi}{\sqrt{2b}} \sqrt{\frac{l_y J_y E_y}{l_x J_x E_x}}, \quad u_n = \frac{k_n a}{2} \quad (19)$$

and we shall seek the solution of our problem in the form

$$w = \sum_{n=1}^{\infty} w_n(x) \sin n\pi \frac{y}{b} \left( +0,5 \right). \quad (20)$$

Having reference to the unknown expansion

$$q = \sum_{n=1}^{\infty} q_n \sin n\pi \left( \frac{y}{b} + 0,5 \right) = \sum_{n=1}^{\infty} \frac{2[1+(-1)^{n+1}]q}{n\pi} \sin n\pi \left( \frac{y}{b} + 0,5 \right). \quad (21)$$

substituting (20) and (21) into Eq. (18), we find for each odd subscript  $n$  an ordinary differential equation

$$w_n^{IV}(x) + 4k_n^4 w_n(x) = \frac{q_n l_y}{E_x J_x}, \quad (22)$$

which is identical with the equation of deflection of a beam on a nonbinding elastic foundation. Its solution is

$$w_n(x) = \frac{q_n l_y}{4k_n^4 E_x J_x} [1 - \varphi(u_n, x)] = \frac{2[1+(-1)^{n+1}] q l_x b^4}{n^5 \pi^5 E_y J_y} [1 - \varphi(u_n, x)], \quad (23)$$

where  $\varphi(u_n, x)$  is the transcendental function expressed by Eq. (10). The sum of the infinite series

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2[1+(-1)^{n+1}] q l_x b^4}{n^5 \pi^5 E_y J_y} \sin \left( \frac{y}{b} + 0,5 \right) &= \\ &= \frac{q l_x b^4}{384 E_y J_y} \left( 5 - 24 \frac{y^2}{b^2} + 16 \frac{y^4}{b^4} \right) \end{aligned} \quad (24)$$

is the elastic curve of the beams of the lattice lined up in the  $y$  direction in the case when no beams have been placed in the other direction.

Using this result we find the final solution of our problem in the form

$$w = \frac{q l_x b^4}{384 E_y J_y} \left\{ 5 - 24 \frac{y^2}{b^2} + 16 \frac{y^4}{b^4} - \frac{768}{\pi^4} \sum_{n=1}^{\infty} \frac{[1 + (-1)^{n+1}]}{n^5} \varphi(u_n, x) \sin n\pi \left( \frac{y}{b} + 0.5 \right) \right\}. \quad (25)$$

In the case when the load on the lattice network is distributed over a part of the flooring area, or is applied in the form of individual concentrated forces, it is expedient to use the method of double series. We introduce the notation

$$\gamma = \frac{a}{b} \sqrt{\frac{B_y}{B_x}}. \quad (26)$$

If we set

$$q_{mn} = \int_{-a/2}^{+a/2} \int_{-b/2}^{+b/2} q(x, y) \sin m\pi \left( \frac{x}{a} + 0.5 \right) \sin n\pi \left( \frac{y}{b} + 0.5 \right) dx dy, \quad (27)$$

then the solution of our problem may be represented in the form of the double series

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4a^3 \gamma q_{mn}}{\pi^4 B_x (m^4 + \gamma^4 n^4)} \sin m\pi \left( \frac{x}{a} + 0.5 \right) \sin n\pi \left( \frac{y}{b} + 0.5 \right). \quad (28)$$

The moments and shearing forces in the lattice's beams are found by differentiating Expressions (25) or (28)

$$\left. \begin{aligned} M_x &= -E_x J_x \frac{\partial^2 w}{\partial x^2}, \quad Q_x = -E_x J_x \frac{\partial^3 w}{\partial x^3}, \\ M_y &= -E_y J_y \frac{\partial^2 w}{\partial y^2}, \quad Q_y = -E_y J_y \frac{\partial^3 w}{\partial y^3}. \end{aligned} \right\} \quad (29)$$

The expansions obtained when solving the problem by the method of simple series converge very well, and if the coordinate axes are lined up so as to conform to the condition  $\gamma \gg 1$ , then, at least when finding deflections and moments, it is possible with accuracy sufficient for practical purposes, to restrict oneself to the retention of only the first term of the series in Expansion (25).

The solution using the two-series method has a substantially poorer convergence, and the number of subscripts  $m$  and  $n$  retained in Expansion (28) should be at least five each to obtain results of acceptable accuracy. However, the great simplicity of these computations in this case substantially facilitates the calculations.

As an illustration of the application of the above methods we consider the design of a square beam lattice with  $a=b=L$  with a beam spacing of  $l_x=l_y=l=\frac{L}{8}$  (Fig. 3). We first consider the case

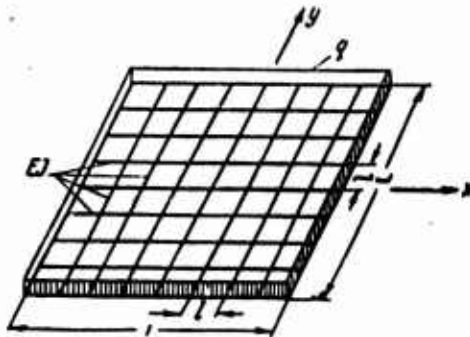


Fig. 3

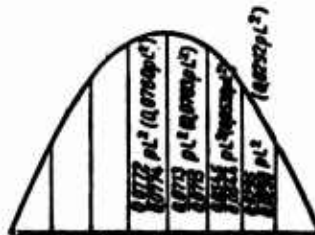


Fig. 4

when the flooring is subjected to a continuous uniform load. Retaining only the first term in Expansion (25), we find the approximate expressions for moments from (29):

$$\left. \begin{aligned} M_x &= \frac{4qly^2}{\pi^2 y^2} \sin \pi \left( \frac{y}{b} + 0,5 \right) \times \\ &\times \frac{2(\operatorname{sh} u_1 \sin u_1 \operatorname{ch} k_1 x \cos k_1 x - \operatorname{ch} u_1 \cos u_1 \operatorname{sh} k_1 x \sin k_1 x)}{\operatorname{ch} 2u_1 + \cos 2u_1}, \\ M_y &= \frac{qlx^2}{8} \left[ 1 - 4 \frac{y^2}{b^2} - \frac{32}{\pi^2} \sin \pi \left( \frac{y}{b} + 0,5 \right) \times \right. \\ &\times \left. \frac{2(\operatorname{sh} u_1 \sin u_1 \operatorname{sh} k_1 x \sin k_1 x + \operatorname{ch} u_1 \cos u_1 \operatorname{ch} k_1 x \cos k_1 x)}{\operatorname{ch} 2u_1 + \cos 2u_1} \right]. \end{aligned} \right\} \quad (30)$$

In our case for  $u_1 = \sqrt{2} \frac{\pi}{4} = 1,1$  we have in the center of the flooring ( $x=0, y=0$ )

$$\begin{aligned} M_{x0} &= \frac{4qL^2}{\pi^2} \cdot \frac{2 \operatorname{sh} u_1 \sin u_1}{\operatorname{ch} 2u_1 + \cos 2u_1} = \frac{4 \cdot 1,2113}{\pi^2 \cdot 2,0266} qL^2 = 0,0772 qL^2, \\ M_{y0} &= \frac{qL^2}{8} \left( 1 - \frac{32}{\pi^2} \cdot \frac{2 \operatorname{ch} u_1 \cos u_1}{\operatorname{ch} 2u_1 + \cos 2u_1} \right) = \left( \frac{1}{8} - \frac{4 \cdot 0,7479}{\pi^2 \cdot 2,0266} \right) qL^2 = 0,0774 qL^2. \end{aligned}$$

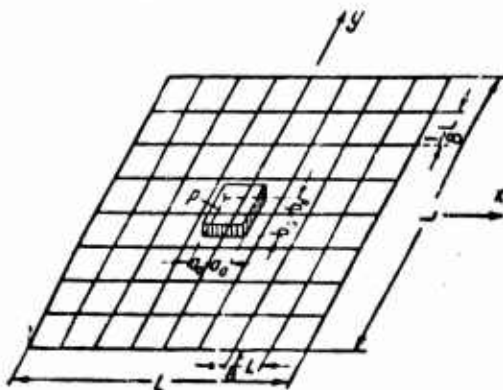


Fig. 5

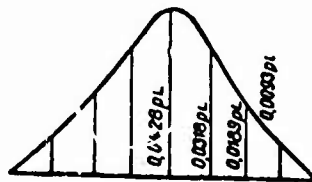


Fig. 6

The difference between these quantities is found to be only 0.3%. Figure 4 shows the moments along axes of the lattice's bars obtained from Formulas (30) (obtained by formulas for  $M_x$  in the numerator and for  $M_y$  in the denominator). For comparison, the parentheses alongside give the corresponding values of coordinates obtained by M.D. Gurari [2] by solving the problem by the method of finite differences. This comparison shows that the results are in satisfactory (with an error not above 2%) agreement.

The maximum and average moments for all the beams of the lattice under study are

$$M_{\max} = 0,0772 q l L^2, \quad M_{cp} = 0,0554 q l L^2,$$

which is, respectively, by 24% higher or 12% lower than the moment  $M = 0,0625 q l L^2$ , which would have acted in beams of each direction, had the load been distributed between them equally and had there been no couplings.

We now consider the case when the lattice is loaded in its center by a concentrated force  $P$ , which we shall represent (Fig. 5) as being uniformly distributed within the rectangle  $2a_0 \times 2b_0$ . Consequently,

$$q = \frac{P}{a_0 b_0} = 64 \frac{P}{L^2}.$$

In this case we find from Formula (27)

$$\begin{aligned} q_{mn} &= \int_{-a_0}^{+a_0} \int_{-b_0}^{+b_0} q \sin m\pi \left( \frac{x}{a} + 0,5 \right) \sin n\pi \left( \frac{y}{b} + 0,5 \right) dx dy = \\ &= \frac{[1 + (-1)^{m+1}][1 + (-1)^{n+1}]}{\pi^2 mn} \sin m\pi \frac{a_0}{a} \sin n\pi \frac{b_0}{b} q. \end{aligned}$$

To obtain the maximum moments in the lattice's beams with an error less than 0.1%, the first seven terms in Expansion (28) must be retained. Upon performing the calculations, we find the following expansion for moments  $M_y$  along the axis of the flooring:

$$\begin{aligned} |M_y|_{y=0} &= \frac{16}{\pi^4} \left[ 0,02528 \sin \pi \left( \frac{x}{a} + 0,5 \right) - \right. \\ &\quad - 0,00435 \sin 3\pi \left( \frac{x}{a} + 0,5 \right) + 0,00170 \sin 5\pi \left( \frac{x}{a} + 0,5 \right) - \\ &\quad - 0,00074 \sin 7\pi \left( \frac{x}{a} + 0,5 \right) + 0,00031 \sin 9\pi \left( \frac{x}{a} + 0,5 \right) - \\ &\quad \left. - 0,00013 \sin 11\pi \left( \frac{x}{a} + 0,5 \right) + 0,00004 \sin 13\pi \left( \frac{x}{a} + 0,5 \right) \right] q l L^2. \end{aligned}$$

For example, in the center of the lattice we have

$$|M_y|_{x=0} = \frac{16}{\pi^4} 0,03255 q/L^3 = 0,00535 q/L^3 = 0,0428 PL.$$

The moment diagram at midspans of the lattice's beams is shown in Fig. 6.

We note that the greatest moment here is only ~34% of the moment which would have arisen in the middle beams of each direction had the load been distributed equally between them, and had the couplings been absent.

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#### Transliterated Symbols

234	$\kappa = k = \text{kanat} = \text{cable}$
234	$\sigma = b = \text{balka} = \text{beam}$
235	$\text{maxc} = \text{maks} = \text{maksimal'nyy} = \text{maximum}$
240	$\text{cp} = \text{sr} = \text{sredniy} = \text{average}$

## APPROXIMATE DESIGN OF PLATES

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The method of plate design discussed here is based on the utilization of ideas of A.P. Filin [1], developed for shells, and results in a simple algorithm of the computational process, which is convenient for electronic digital computer [EDC] (3UBM) programming. The main advantage of this method is the possibility to use it for any plate shape, any law governing the variation in its thickness and for any load. Here the problem reduces to considering a system of linear equations, which can be solved on an EDC using standard programs.

### §1. The Design Scheme

The primary geometric assumptions used in designing plates are, as is known, similar to those used in the design of beams, namely: a) the deflections in all the points are assumed to be normal to the initial plane of the plate and b) after deformation the normal in any point of the middle surface remains straight within the limits of the plate's thickness (analogy to the hypothesis of plane sections). These assumptions make it natural to reduce the plate to a system of intersecting beams and enable one to use principals from the strength of materials in constructing an approximate method for designing plates. A remark on this idea may be found as early as in [2], but there it is not worked out in detail.

The design scheme of the plate being here considered is shown in Fig. 1. The plate is here replaced by two layers of beams which shall, by convention, be called longitudinal and transverse. The longitudinal beams form the upper and the transverse beams make up the lower layer. This division into two layers is needed primarily for convenience and clarity in describing the design scheme; in general it is possible to forgo this assumption and to assume that the two layers of beams seemingly penetrate into one another.

Under the action of external forces three mutual forces are produced in each center of intersection of two beams: 1) vertical forces  $S_{m,n}$ , where  $m$  is the number of the longitudinal row and  $n$  is the number of the transverse row; 2) moments  $M_{m,n}$  in the vertical plane  $XOZ$ ; 3) moments  $M_{n,m}$  in the vertical plane  $YOZ$ .

The above forces are for each of the two beams intersecting in the given point  $m, n$  external forces; here forces  $S_{m,n}$  produce flexure in the beams, while moment  $M_{m,n}$  produces flexure in longitudinal beam  $m$  and torsion in transverse beam  $n$  and, conversely, moment  $M_{n,m}$  produces flexure in transverse beam  $n$  and torsion in longitudinal beam  $m$ . (The beams are numbered starting from coordinate axes  $X$  and  $Y$ , respectively.)

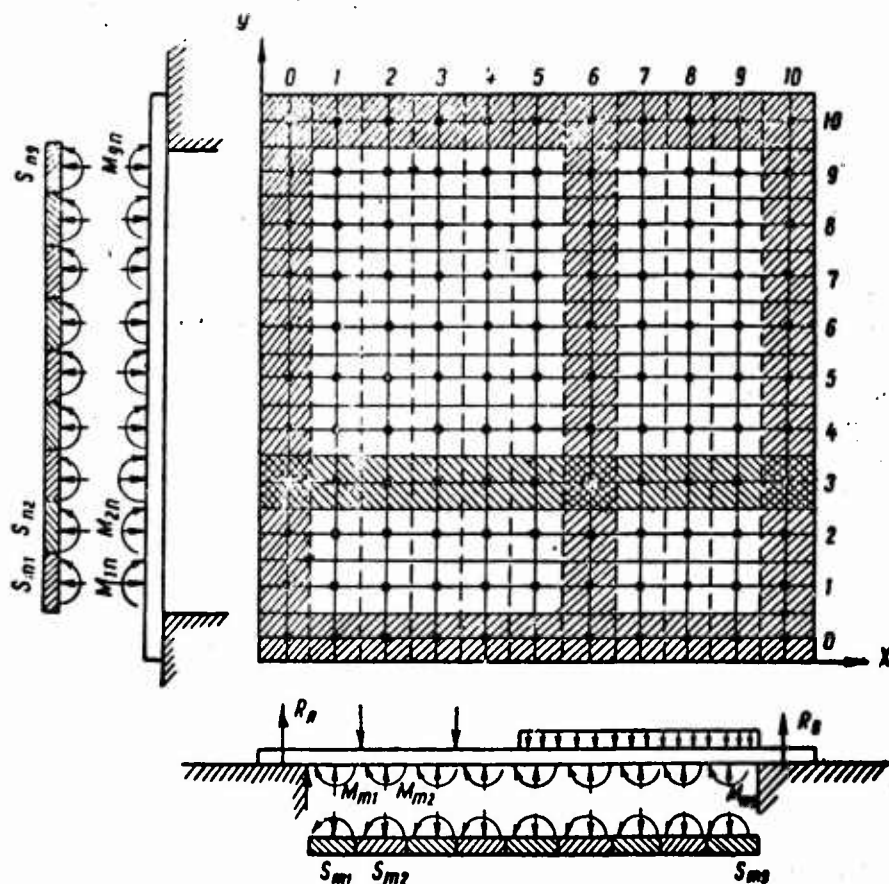


Fig. 1

If the number of longitudinal beams is  $u$ , while that of transverse beams is  $v$ , then the total number of unknowns is

$$N = 3uv. \quad (1)$$

Thus, for the case shown in Fig. 1, the number of unknowns is

$$N = 3 \cdot 9 \cdot 9 = 243.$$

## §2. Canonical Equations

To set up the necessary  $N$  equations we use the condition of compatibility of strains, namely:

- 1) the deflections of the longitudinal and transverse beams

at point  $m, n$  are equal

$$w_{m,n} = w_{n,m}; \quad (2)$$

2) the angle of rotation of the longitudinal beam is equal to the angle of twist of the transverse beam

$$\theta_{m,n} = \varphi_{n,m}; \quad (3)$$

3) the angle of twist of the longitudinal beam is equal to the angle of rotation of the transverse beam

$$\varphi_{m,n} = \theta_{n,m}. \quad (4)$$

Each of these six translations in the given point may be expressed as a function of the acting load and of all the interaction forces applied to the beam the translation of which we seek.

Henceforth, in order to standardize the form of calculations, we agree to refer the entire load to the upper beam layer. The longitudinal beams are thus subjected to the acting load, as well as to forces  $S_{m,n}$ ,  $M_{m,n}$  and  $M_{n,m}$ ; while the transverse beams are subjected only to forces  $S_{n,m}$ ,  $M_{n,m}$  and  $M_{m,n}$ .

To set up developed equations for the given point  $m, n$ , it would have been sufficient to write the expressions for deflections, angles of rotation and angles of twist of the longitudinal and transverse beams at point  $m, n$  and to then substitute into Eqs. (2)-(4).

The free terms of equations will here be obtained from the expressions of translations of the longitudinal beams, since the free terms depend solely on the acting load, and do not depend on the forces of interaction between the beams.

Each of the three equations which are obtained by equating the translations in the given point (node center)  $m, n$  will contain terms containing unknown forces of interaction applied only to the two beams intersecting at point  $m, n$ .

The problem thus reduces to evaluating a matrix, and the coefficients of the unknown and free terms.

It is precisely this part of the problem which is regarded as the basic topic of this paper. As will be seen below, the expressions for coefficients for the unknowns will result in a limited number of standard and quite simple formulas, the general form of which does not depend on the particular conditions of a given problem.

Formulas for the free terms cannot, naturally, be standard, by virtue of the possible variety of loads. However, even here, as will be shown, the calculations reduce to performing the most elementary operations.

We now consider the obtaining of equations for the case of a rectangular plate freely supported along its contour (see Fig. 1).

Let it be required to set up for this case an equation such as (2)

$$w_{m,n} = w_{n,m}. \quad (5)$$

Deflection  $w_{m,n}$  is determined at the point of intersection of longitudinal beam  $m$  with transverse beam  $n$ . This deflection should be expressed as a function only of the specified acting forces and of the forces of interaction acting on the longitudinal beam. In general this deflection is defined as<sup>1</sup>:

$$w_{m,n} = \sum_{l_1=1}^{l_1=v} \alpha_{ml_1} M_{ml_1} + \sum_{l_1=1}^{l_1=v} \beta_{ml_1} S_{ml_1} + \Omega_{m,n}. \quad (6)$$

Here the first summation sign contains the products of all the moments applied to beam  $m$  in the  $XOZ$  plane and of the, as yet unknown, coefficients  $\alpha_{ml_1}$  corresponding to them. The second summation sign contains the products of all the vertical forces applied to beam  $m$  by the unknown coefficients  $\beta_{ml_1}$ . Finally, quantity  $\Omega_{m,n}$  is the deflection of beam  $m$  due to the external acting force acting on this beam.

In an entirely similar manner deflection  $w_{n,m}$  of the transverse beam in the same point  $m, n$  is defined as

$$w_{n,m} = - \left[ \sum_{l_1=1}^{l_1=u} \alpha_{nl_1} M_{nl_1} + \sum_{l_1=1}^{l_1=u} \beta_{nl_1} S_{nl_1} \right]. \quad (7)$$

Unlike Expression (6) no third term exists here since the entire acting load of the plate is transferred solely to the longitudinal beams.

Expression (7) is written with a minus sign because here the signs of all the interaction forces are opposite relative to the corresponding signs of forces in Eq. (6).

Substituting Eqs. (6) and (7) into (5), we get an equation of the first type in the following form

$$\sum_{l_1=1}^{l_1=v} \alpha_{ml_1} M_{ml_1} + \sum_{l_1=1}^{l_1=v} \beta_{ml_1} S_{ml_1} + \sum_{l_1=1}^{l_1=u} \alpha_{nl_1} + \sum_{l_1=1}^{l_1=u} \beta_{nl_1} S_{nl_1} + \Omega_{m,n} = 0. \quad (8)$$

Proceeding in a similar manner, it is possible to obtain an equation of the second type (3) in the form

$$\sum_{l_1=1}^{l_1=v} \gamma_{ml_1} M_{ml_1} + \sum_{l_1=1}^{l_1=v} \delta_{ml_1} S_{ml_1} + \sum_{l_1=1}^{l_1=u} \varepsilon_{nl_1} M_{nl_1} + \Psi_{m,n} = 0. \quad (9)$$

We note that products  $\varepsilon_{nl_1} M_{nl_1}$  define the angles of twist of transverse beam  $n$  in point  $m, n$ . Quantities  $M_{nl_1}$  are moments of interaction, applied to transverse beam  $n$  in planes  $XOZ$ , i.e., moments producing torsion in these beams.

Equation (9) contains one term less than Eq. (8), since the angle of twist depends only on the moments of interaction and does not depend on the forces of interaction.

Finally, the equation of the third type (4) is expressed as

$$\sum_{i_1=1}^{i_1=v} e_{mi_1} \bar{M}_{mi_1} + \sum_{i_1=1}^{i_1=u} \gamma_{ni_1} M_{ni_1} + \sum_{i_1=1}^{i_1=u} \delta_{ni_1} S_{ni_1} = 0. \quad (10)$$

This equation does not contain the free term, since only twisting is considered in the longitudinal beam, while the external acting load as such does not produce torsion in the beams. As we have agreed, the external load does not act at all on the transverse beams.

The horizontal bar over the expression of the moment means that this moment acts in the vertical plane, perpendicular to the vertical plane passing through the axis of the beam in question, i.e., produces torsion in this beam.

### §3. Coefficients of Equations

Coefficients  $\alpha_{mi_1}$  and  $\beta_{mi_1}$  in Eq. (6) are the deflections at point  $m, n$  produced by the unit moment  $M_{mi_1}=1$  and unit force  $S_{mi_1}=1$ , applied at point  $mi_1$ . Consequently, quantities  $\alpha_{mi_1}$  and  $\beta_{mi_1}$  may be regarded as ordinates, at point  $mi_1$ , of the influence lines of deflection due to  $M=1$  and  $S=1$ , and to obtain the magnitudes of  $\alpha_{mi_1}$  and  $\beta_{mi_1}$  it would have sufficed to construct these influence lines or to write their equations. Let us now turn to setting up these equations.

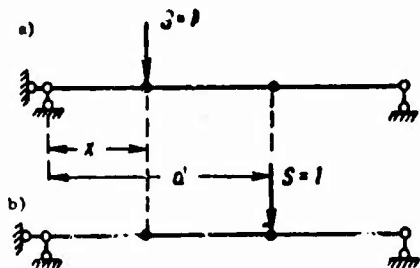


Fig. 2

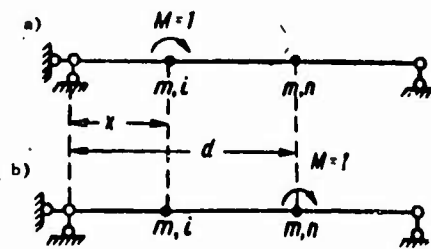


Fig. 3

We take into account the following circumstance: in considering the equations of the problem with  $w_{m,n}=w_{n,m}$  point  $m, n$  will be specified, while points  $mi$  correspond to points of application of force  $S_i=1$  (Fig. 2a) or moment  $M=1$  (Fig. 3a). In this case it is expedient to use the principle of reciprocal translations: instead of determining the deflection at point  $m, n$  produced by force  $S=1$  or moment  $M=1$  applied at point  $mi$ , to seek the same deflection at point  $mi$  produced by  $S=1$  or  $M=1$  applied at point  $m, n$ ; in other words, it is necessary to write the expression of the deflection due to unit moment or unit force applied at point  $m, n$  (Figs. 2c, 3c).

It is known that at the left segment (Fig. 2b)

$$\theta_{1P} = \frac{S(1-\mu^2)}{6EJ_y l} [-3(l-a)x_1^2 - (2l^2 a + 3la^2 + a^3)]; \quad (11)$$

$$w_{1P} = \frac{S(1-\mu^2)}{6EJ_y l} [(l-a)x_1^3 - (2l^2 a + 3la^2 + a^3)x_1]. \quad (12)$$

At the right segment (Fig. 2c)

$$\theta_{2P} = \frac{S(1-\mu^2)}{6EJ_y l} (-3ax_2^2 + 2a^2 x - 2l^2 a - a^3); \quad (13)$$

$$w_{2P} = \frac{S(1-\mu^2)}{6EJ_y l} [-ax_2^3 - 3alx_2^2 + (2l^3 - l^2 a - a^2)x_2 + la^3]. \quad (14)$$

Further, it is known that at the left segment (Fig. 3b)

$$\theta_{1M} = \frac{M(1-\mu^2)}{6EJ_y l} (-3x_1^2 - 2l^2 + 6la - 3a^2); \quad (15)$$

$$w_{1M} = \frac{M(1-\mu^2)}{6EJ_y l} (-x_1^3 + 2l^2 x_1 + 6la - 3a^2)x_1. \quad (16)$$

At the right segment (Fig. 3c)

$$\theta_{2M} = \frac{M(1-\mu^2)}{6EJ_y l} (-x_2^2 + 6lx_2 - 2l^2 - 3a^2); \quad (17)$$

$$w_{2M} = \frac{M(1-\mu^2)}{6EJ_y l} [x_2^3 + 3lx_2^2 - (2l^3 + 3a^2)x_2 + 3a^2l]. \quad (18)$$

We now write formulas for the angle of twist for a beam loaded in the manner shown in Fig. 4a. For this we shall also use the principle of reciprocal translations and shall use the scheme of Fig. 4b, i.e., we shall seek the expression of the angle of twist at point  $m_i$  produced by unit twisting moment applied at point  $m, n$ . We take into account the fact that the initial angle of twist (in point A) is zero. This follows from the fact that the plate does not deflect along the contour; in this case the angle of rotation of the supporting beam perpendicular to the axis of the given beam is equal to zero and, consequently, the support angle of twist of the given beams is also zero.

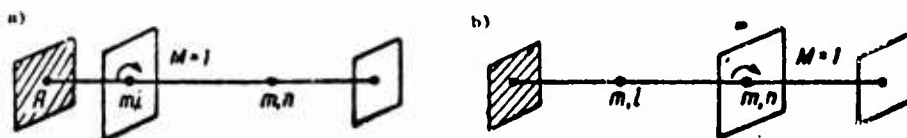


Fig. 4

The reactive twisting moment is determined from the condition that the angle of twist at the right support is also zero

$$\varphi_{mB} = 0,$$

then

$$\frac{1}{GJ_x} [M_r l + M_1 (l-a)] = 0,$$

whence, making use of the fact that  $\frac{1}{GJ_x} \neq 0$ ,

$$M_r = -M_1 \frac{l-a}{l}. \quad (19)$$

Now it is possible to also determine the angles of twist at both segments of the beam.

At the left segment

$$\varphi_1 = -\frac{M_1}{GJ_x} \cdot \frac{a}{l} x_1. \quad (20)$$

At the right segment

$$\varphi_2 = \frac{M_1}{GJ_x} \left( \frac{a}{l} x_2 + a \right). \quad (21)$$

The free terms of Eqs. (8)-(10) are the translations due to the external acting force applied to the given beam  $m$ ; consequently,

$$\begin{aligned} \Omega &= w_{m,n}^{(P)}; \\ \Psi &= \theta_{m,n}^{(P)}. \end{aligned}$$

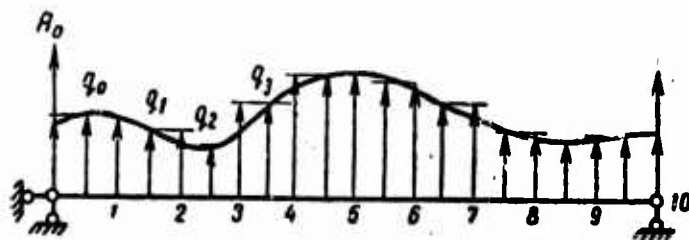
Calculation of deflections  $w_{m,n}^{(P)}$  and angles of twist  $\theta_{m,n}^{(P)}$  is not as simple as calculating the coefficients of the unknowns, since the external acting load may be arbitrary. Nevertheless, even here the calculations may be reduced to a regular process.

Let for longitudinal beam  $m$  be known the external acting load  $q(x)$  acting in the plane of the geometric axis of this beam (Fig. 5).<sup>2</sup> In each of its junctions  $1, 2, 3, \dots, 9$  it is necessary to determine the angle of rotation and deflection due to this load. These translations will enter Formulas (8) and (9).

$$\begin{aligned} 0 &= 0_0 + \frac{1-\mu^2}{EJ_y} \left[ \sum \frac{M_m}{1!} + \sum \frac{S_s^2}{2!} + \sum \frac{q(k_n^3 - k_k^3)}{3!} \right]; \\ w &= w_0 + \theta_0 x + \frac{1-\mu^2}{EJ_y} \left[ \sum \frac{M_m x}{2!} + \sum \frac{S_s^3}{3!} + \sum \frac{q(k_n^4 - k_k^4)}{4!} \right]; \\ \varphi &= \varphi_0 + \frac{1}{GJ_x} \sum M_1 m. \end{aligned}$$

Fig. 5

These translations may be determined most simply by moving from node to node, for which it is necessary to use the known formulas of the method of initial parameters



For the continuous load assumed by us it is possible, by moving in sequence from node to node, to write expressions for the internal forces and translations in node  $n$ , expressing them in terms of these translations and forces in node  $(n-1)$  and in terms of the external-load application rate  $q_{n-1}$ , using the formulas

$$Q_n = Q_{n-1} + q_{n-1}d; \quad (22)$$

$$M_n = M_{n-1} + Q_{n-1}d + \frac{q_{n-1}d^2}{2}; \quad (23)$$

$$\theta_n = \theta_{n-1} + \frac{1-\mu^2}{EJ_y} \left( M_{n-1}d + \frac{Q_{n-1}d^2}{2} + \frac{q_{n-1}d^3}{6} \right); \quad (24)$$

$$w_n = w_{n-1} + \theta_{n-1}d + \frac{1-\mu^2}{EJ_y} \left( \frac{M_{n-1}d^2}{2} + \frac{Q_{n-1}d^3}{6} + \frac{q_{n-1}d^4}{24} \right), \quad (25)$$

where  $d$  is the distance between nodes.

We note that the initial parameters must be predetermined. For our scheme of a plate freely supported along the contour,  $M_0=0$ , and  $W_0=0$ ;  $Q_0=A_0$  (Fig. 5).

As to the initial angle of rotation  $\theta_0$ , it should be determined from the condition that the deflection at the right support is zero

$$w_B = 0.$$

In the recurrent method of calculations described here it would be most convenient to retain in all the calculations  $\theta_0$  in the symbol form and, only when reaching the right support point, to equate to zero the equation of deflection for it.

It is recommended that the rate of application  $q_{n-1}$  of the load in Formulas (22)-(25) be thought of as the mean ordinate of the load curve at the segment between nodes  $n-1$  and  $n$  (see Fig. 5).

The final values of the shearing forces and bending (twisting) moments in the longitudinal and transverse beams for each node may be determined after solving the solution to the system of equations, when interaction forces  $M_{m,n}$  and  $S_{m,n}$  are found.

The shearing force will be determined as the sum of two shearing forces at the given point of the beam's axis due to the external active force and to interaction forces  $S_{m,n}$ . This last operation is most conveniently performed also recurrently, moving from node to node; then<sup>3</sup>

$$Q_{m,n} = Q_{m,n-1} + S_{m,n-1}.$$

Similarly, for the bending moment

$$\overline{M}_{m,n}^{(n)} = M_{m,n-1}^{(n)} + M_{m,n-1} + Q_{n-1} d.$$

The twisting moment in the longitudinal beam is equal to the bending moment of the transverse beam at the given point and vice versa

$$M_{m,n}^{(n)} = M_{n,m}^{(n)}.$$

The total internal forces are

$$Q_{m,n} = Q_{m,n}^{(P)} - \overline{Q}_{m,n};$$

$$M_{m,n}^{(n)} = M_{m,n}^{(P)} + \overline{M}_{m,n}^{(n)}.$$

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#### Footnotes

- 245 <sup>1</sup>In Formulas (6) and (7), quantities  $i_1$  and  $i_2$  will denote the numbers of the longitudinal ( $i_1$ ) and transverse ( $i_2$ ) beams, starting with 0 at the supports adjoining the coordinate axes.
- 248 <sup>2</sup>We reduce the load to the longitudinal axis of the beam in a simplified manner, irrespective of the fact whether it is distributed symmetrically over the width of the beam. This involves a certain degree of approximation, which will have the lesser effect, the smaller the beam width selected by us.
- 249 <sup>3</sup>Below  $\overline{Q}$  and  $\overline{M}_{m,n}^{(n)}$  denote internal forces, which take into account only the forces of interaction.

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Transliterated Symbols

248      κ = k = krutyashchiy = twisting  
249      н = n = nachalo = start  
249      κ = k = konets = end  
250      и = i = izgib = bending

# DESIGN OF VARIABLE-THICKNESS PLATES USING AN ELECTRONIC COMPUTER

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## §1. Introduction

The present article considers the stability and transverse bending of a variable-thickness plate, and also contains an analysis of the effect of certain cross-sectional shapes of a plate on the magnitude of the critical load. The solution is presented by the numerical method in a form making it possible to fully mechanize the computations, which is obtained for using a method suggested by Professor A.F. Smirnov [2], [3]. This method is based on a solution of linear differential equations by means of an integral matrix which was developed by Professor A.F. Smirnov and presented in his book [1]. The use of this solution makes it possible, after appropriate transformations, to write the above differential equation in vector form and to reduce the problem almost completely to performing operations of linear algebra, which is not difficult when using computers.

The solution of the problem presented below was performed entirely on the "Ural-2" electronic computer. In setting up the program use was made of a standard subprogram for solving problems of linear algebra, developed by Docent N.N. Shaposhnikov.

## §2. Basic Equations

We consider a plate (Fig. 1) of variable thickness  $h=h(y)$ , compressed at sides  $x=0$  and  $x=a$  by forces  $N_x^*(y)$  (per unit length) in the direction of the  $X$ -axis, and loaded by a transverse load  $q=q(xy)$ . The differential equation of flexure for a plate in this case has the form

$$D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^2 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + 2 \frac{\partial D}{\partial y} \left( \frac{\partial^3 w}{\partial y^3} + \frac{\partial^2 w}{\partial y \partial x^2} \right) + \frac{\partial^2 D}{\partial y^2} \left( \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) + N_x^* \frac{\partial^2 w}{\partial x^2} + N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} - q = 0, \quad (1)$$

where

$\mu$  is Poisson's ratio;

$D$  is the cylindrical rigidity;

$N_x$  and  $N_y$  are forces per unit length acting in the middle plane of the plate.

Let the plate be freely supported along sides  $x=0$  and  $x=a$ , with the support conditions of the other two sides arbitrary. Then it

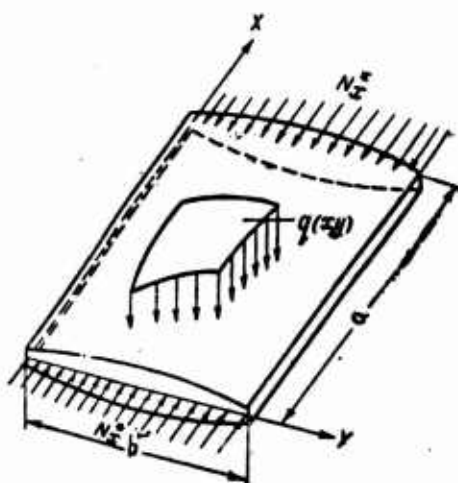


Fig. 1

is possible to seek the approximate solution of Eq. (1) in the form

$$w = \sum_1^n \sin \frac{m\pi x}{a} f_m(y) \quad (2)$$

when  $m = 1, 2, 3, \dots, n$ . Substituting (2) into (1), having first represented the load in the form

$$q(x, y) = \sum_1^n \sin \frac{m\pi x}{a} F_m(y), \quad (3)$$

we get on the assumption that  $N_{xy} = 0$ .

$$\begin{aligned} & \sum_1^n \sin \frac{m\pi x}{a} \left[ D \left( f_m^{IV} - 2 \frac{m^2 \pi^2}{a^2} f_m'' + \frac{m^4 \pi^4}{a^4} f_m \right) + 2 \frac{\partial D}{\partial y} \left( f_m''' - \frac{m^2 \pi^2}{a^2} f_m' \right) + \frac{\partial^2 D}{\partial y^2} \left( f_m'' - \mu \frac{m^2 \pi^2}{a^2} f_m' \right) - \right. \\ & \left. - \frac{m^2 \pi^2}{a^2} N_x f_m + N_y f_m \right] = \sum_1^n \sin \frac{m\pi x}{a} F_m \end{aligned} \quad (4)$$

for  $m = 1, 2, 3, \dots, n$ .

In order for Equality (4) to apply, it is required to satisfy the condition

$$\begin{aligned} & D \left( f_m^{IV} - 2 \frac{m^2 \pi^2}{a^2} f_m'' + \frac{m^4 \pi^4}{a^4} f_m \right) + 2 \frac{\partial D}{\partial y} \left( f_m''' - \frac{m^2 \pi^2}{a^2} f_m' \right) + \\ & + \frac{\partial^2 D}{\partial y^2} \left( f_m'' - \mu \frac{m^2 \pi^2}{a^2} f_m' \right) - \frac{m^2 \pi^2}{a^2} N_x f_m + N_y f_m = F_m \end{aligned} \quad (5)$$

for  $m = 1, 2, 3, \dots, n$ .

Upon finding  $f_m$  from Eq. (5) and substituting it into (2), we find the sought solution of Eq. (1).

### §3. Designing the Plate for Stability

We consider a plate subjected solely to compressive forces  $N_x^*(y)$ .

It can be shown that the solution of Eq. (1) in this case will have the form

$$w = \sin \frac{m\pi x}{a} f(y), \quad (6)$$

where  $m$  is a positive integral number. Then to determine  $f(y)$  we get

$$\begin{aligned} & D \left( f^{IV} - 2 \frac{m^2 \pi^2}{a^2} f'' + \frac{m^4 \pi^4}{a^4} f \right) + 2 \frac{\partial D}{\partial y} \left( f''' + \frac{m^2 \pi^2}{a^2} f' \right) + \\ & + \frac{\partial^2 D}{\partial y^2} \left( f'' - \mu \frac{m^2 \pi^2}{a^2} f' \right) - \frac{m^2 \pi^2}{a^2} N_x f'' + N_y f = 0. \end{aligned} \quad (7)$$

Determination of  $N_x$  and  $N_y$  contained in (7) is in general quite difficult. Let us consider a number of cases when the finding of stresses may be simplified:

$$a) \quad \sigma_x = \frac{N_x^*}{h} = \text{const.}$$

In this case

$$N_x = N_x^*; N_y = N_{xy} = 0;$$

$$b) \quad N_x^* = \text{const.}$$

In the case when the plate's cross section has two axes of symmetry it is possible, with a high degree of accuracy, to use for the stresses the solution given by strength of materials, i.e.,

$$N_x = N_x^*; N_y = N_{xy} = 0;$$

$$u) \quad h = \text{const} \text{ и } N_x^* -$$

varies linearly. Then also

$$N_x = N_x^*; N_y = N_{xy} = 0.$$

Solving Eq. (7) by the method suggested by Professor A.F. Smirnov [2, 3] and using the notation in [2, 3], it is possible to reduce Eq. (7) to the form

$$(C - \lambda E) f^{IV} = 0, \quad (8)$$

where

$$C = L^{-1} L_0;$$

$$L = \bar{E} - 2 \frac{m^2 \pi^2}{a^2} C_2 + \frac{m^4 \pi^4}{a^4} C_0 + 2G_1 \left( C_3 - \frac{m^2 \pi^2}{a^2} C_1 \right) + G_2 \left( C_2 - \mu \frac{m^2 \pi^2}{a^2} C_0 \right);$$

$$L_0 = \frac{m^2 \pi^2}{a^2} C_N G_3 C_0.$$

Here  $\frac{1}{\lambda} = \frac{N_{x0} b^3}{D_0};$

$C_N$  is the diagonal matrix

$$C_N = \begin{bmatrix} \frac{N_{x1}}{N_{x0}} & & \\ & \ddots & \\ & & \frac{N_{xn}}{N_{x0}} \end{bmatrix}$$

Matrices  $C_i$  are obtained from the equalities

$$\left. \begin{aligned} C_0 &= \sigma f_0 + \tau f_0' + \frac{\tau^2}{2!} f_0'' + \frac{\tau^3}{3!} f_0''' + \Omega^4 \bar{f}^{IV}; \\ C_1 &= \sigma f_0' + \tau f_0'' + \frac{\tau^2}{2!} f_0''' + \Omega^3 \bar{f}^{IV}; \\ C_2 &= \sigma f_0'' + \tau f_0''' + \Omega^2 \bar{f}^{IV}; \\ C_3 &= \sigma f_0''' + \Omega \bar{f}^{IV}. \end{aligned} \right\} \quad (9)$$

Setting

$$|C - \lambda E| = 0,$$

we get a nontrivial solution of Eq. (8), whence

$$N_{xkp} = \frac{D_0}{\lambda b^2}.$$

The initial parameters  $f_0$ ,  $f'_0$ ,  $f''_0$  and  $f'''_0$ , which are contained in Equalities (9) may be obtained from boundary conditions. We shall assume that the plate is elastically supported and elastically clamped along sides  $y=0$  and  $y=b$ . Then the boundary conditions have the form

$$\left. \begin{aligned} \frac{\partial w}{\partial y} K_1 + D_0 \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} &= 0 \\ \frac{\partial^4 w}{\partial x^4} K_2 + D_0 \left[ \frac{\partial^2 w}{\partial y^2} + (2-\mu) \frac{\partial^2 w}{\partial y \partial x^2} \right] &= 0 \end{aligned} \right\} \text{ for } y=0$$

$$\left. \begin{aligned} \frac{\partial w}{\partial y} K_3 + D_n \left( \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) &= 0 \\ \frac{\partial^4 w}{\partial x^4} K_4 + D_n \left( \frac{\partial^2 w}{\partial y^2} + (2-\mu) \frac{\partial^2 w}{\partial y \partial x^2} \right) &= 0 \end{aligned} \right\} \text{ for } y=b,$$
(10)

where  $K_1$  and  $K_3$  are the proportionality factors between the angle of rotation and the twisting moment at sides  $y=0$  and  $y=b$ , respectively, while  $K_2$  and  $K_4$  are the proportionality factors between the shearing force and translation at sides  $y=0$  and  $y=b$ . Transforming (10) according to the method used in these calculations, we get for the determination of  $f_0^{(i)}$ , in the case of constant  $K$  a system of equations of the form

$$\left. \begin{aligned} \alpha_1 f_0 + \alpha_2 f'_0 + f''_0 &= 0; \\ \alpha_3 f_0 + \alpha_4 f'_0 + f''_0 &= 0; \\ \beta_1 f_0 + (\beta_1 + \beta_2) f'_0 + \left( \frac{\beta_1}{2} + \beta_2 + 1 \right) f''_0 + \left( \frac{\beta_1}{6} + \frac{\beta_2}{2} + 1 \right) f'''_0 &= \\ &= -(\beta_1 \omega^4 + \beta_2 \omega^3 + \omega^2) \bar{f}^{IV}; \\ \beta_3 f_0 + (\beta_3 + \beta_4) f'_0 + \left( \frac{\beta_3}{2} + \beta_4 \right) f''_0 + \left( \frac{\beta_3}{6} + \frac{\beta_4}{2} + 1 \right) f'''_0 &= \\ &= -(\beta_3 \omega^4 + \beta_4 \omega^3 + \omega) \bar{f}^{IV}. \end{aligned} \right\} \quad (11)$$

Here  $\omega^k$  is the bottom row of matrix  $\Omega^k$ ; the coefficients are

$$\alpha_1 = -\mu \frac{m^2 \pi^4}{a^2}; \quad \alpha_2 = \frac{K_1 b}{D_0}; \quad \alpha_3 = \frac{m^4 \pi^4}{a^4} - \frac{K_2}{D_0 b}; \quad \alpha_4 = (\mu - 2) \frac{m^2 \pi^2}{a^2};$$

$$\beta_1 = \alpha_1; \quad \beta_2 = \frac{K_3 b}{D_n}; \quad \beta_3 = \frac{m^4 \pi^4}{a^4} - \frac{K_4}{D_n b}; \quad \beta_4 = \alpha_4.$$

A substantial effect on the accuracy of solution obtained by the numerical method used is the number of parts into which the plate width is broken up. To determine the required number of divisions we now consider a solution for a constant-thickness plate when the plate width  $b$  is divided into 5 and 10 parts.

This problem is solved by using an integral matrix, based on approximating the function and its derivative by means of Lagrange polynomials. In this case, as is known, the use of an integral matrix of an order higher than six may not result in a higher accuracy than when using lower-order matrices, which follows from

the properties of Lagrange polynomials. To avoid this, we apply the following operation to plates with symmetrical cross section. We break up the plate along its width into two parts, having first broken up the load into symmetrical and skewed-symmetrical, and we shall seek a solution for one half of the plate, upon breaking it up along its width into five parts. Here it is necessary to assume for the symmetrical load along the dividing line that the plate is rigidly clamped with the edge free to move, while for the skewed-symmetrical load it is assumed that the edge is freely supported. The table presents results of calculating the critical load for a constant-thickness square plate with a different number of divisions, as well as values presented in the book by S.P. Timoshenko [4].

Value of Coefficients  $k \left( N_{кр} = k \frac{\pi^2 D_0}{a^3} \right)$

1 Граничные условия	2 Численный метод		3 По Тим- мошенко
	деление на 5 частей 4	деление на 10 ча- стей 5	
Края $y=0$ и $y=b$ 6 свободно оперты	4,0004	4,0000	4
Края $y=0$ и $y=b$ 7 заделаны	7,6321	7,6913	7,69
Края $y=0$ и $y=b$ 8 свободны	0,9523	0,9523	—

- 1) Boundary conditions
- 2) Numerical method
- 3) According to Timoshenko
- 4) Division into 5 parts
- 5) Division into 10 parts
- 6) Edges ... and ... freely supported
- 7) Edges ... and ... clamped
- 8) Edges ... and ... free.

As can be seen from the table, dividing the plate into five parts already ensures sufficient accuracy.

Below are presented critical loads calculated for plates with different cross sections. Here consideration is given to plates with the same cross-sectional areas, which makes it possible to make a certain estimate of the efficiency of a given cross-sectional shape.

a) Plate with a wedge-like cross section (Fig. 2a) subjected along sides  $x=0$  and  $x=a$  to a variable load  $N_x^*(y)$ , so that

$$\sigma_x = \frac{N_x^*}{h} = \text{const.}$$

In the given case we should set in Eq. (8)  $C_N = C_h$ , where

$$C_h = \left\| \frac{h_1}{h_0} \cdot \dots \cdot \frac{h_1}{h_0} \right\|$$

Figure 2b shows the variation of critical stresses  $\sigma_{xkr}$  for the given plate as a function of ratio  $\frac{h_n}{h_0}$ . It follows from the graph (Fig. 2b) that maximum stresses  $\sigma_{xkr}$  will exist in a constant thickness plate, when plates of the same volume are compared.

b) Plate with symmetrical cross section subjected to the constant load

$$N_x^* = \text{const.}$$

The solution for the case under study is obtained by assuming in Eq. (8)

$$C_N = E.$$

The results of calculations for a plate with a cross section in the form of two combined wedges (Fig. 3a and 3b) are shown in the graph (Fig. 3c).

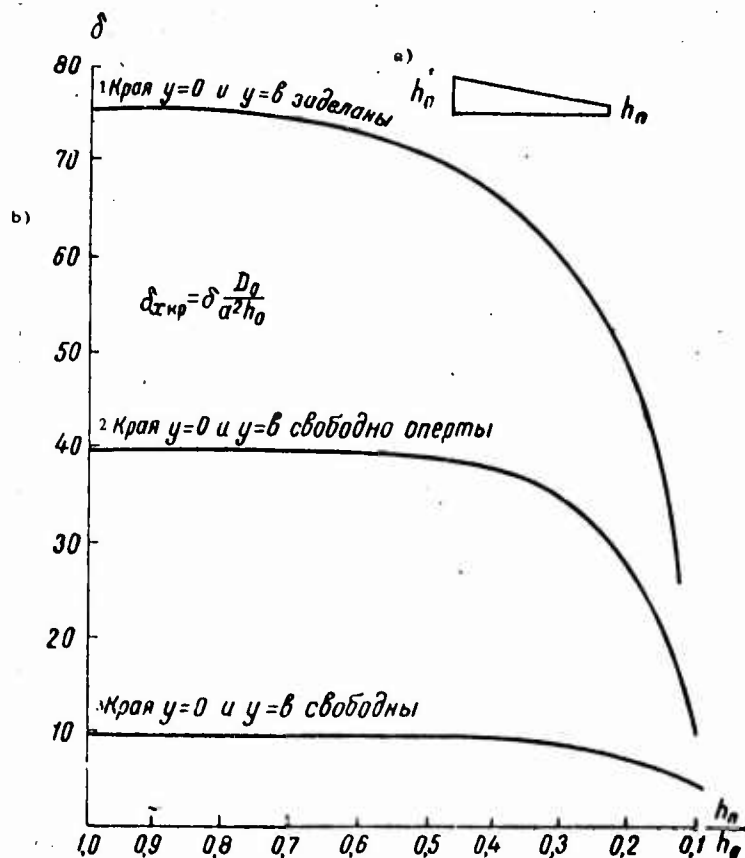


Fig. 2. 1) Edges ... and ... are clamped; 2) edges ... and ... freely supported; 3) edges ... and ... free.

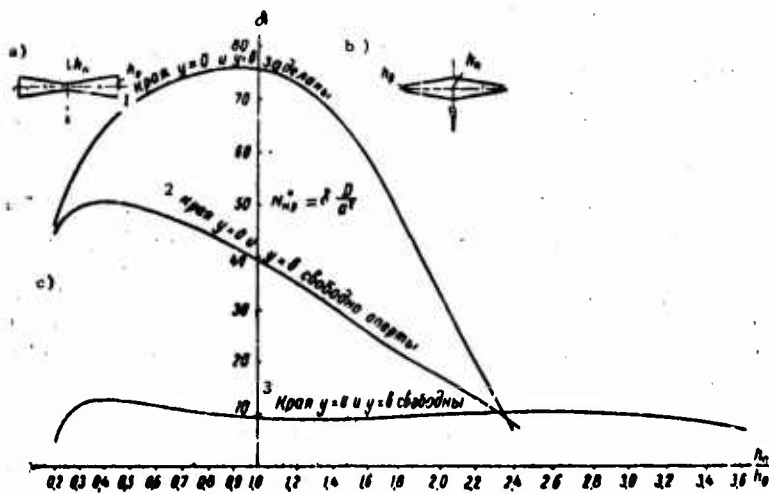


Fig. 3. 1) Edges ... and ... clamped; 2) edges ... and ... freely supported; 3) edges ... and ... free.

It may be concluded from the graph (Fig. 3c) that the most efficient cross-sectional shape for the majority of the cases under consideration is that with maximum material concentration at the edges.

#### §4. Transverse Flexure of a Variable-Thickness Plate

The differential equation of flexure for a plate for the case being considered is easily obtained from (1) by setting

$$N_x = N_y = N_{x,y} = 0.$$

We shall seek the solution of (1) in the form of (2), when  $f_m$  may be obtained from Eq. (5), where it should be assumed that  $N_x = N_y = 0$ . Solving (5) by the numerical method used in the preceding section, we get, after transformations, Eq. (5) in the vector form

$$L_m \bar{f}^{IV} = G_3 \bar{F}_m. \quad (12)$$

Here

$$L_m = E - 2 \frac{m^2 \pi^2}{a^2} C_2 + \frac{m^4 \pi^4}{a^4} C_0 + 2G_1 \left( C_3 - \frac{m^2 \pi^2}{a^2} C_1 \right) + G_2 \left( C_2 - \mu \frac{m^2 \pi^2}{a^2} C_0 \right); \quad (13)$$

$\bar{F}_m$  is a load vector, consisting of the values of the function in points  $0, 1, 2, \dots, n$ ;

$$\bar{F}_m = \begin{pmatrix} F(0) \\ F\left(\frac{b}{n}\right) \\ F\left(2\frac{b}{n}\right) \\ \vdots \\ F(2b) \end{pmatrix}$$

Matrices  $C_i$  contained in (13) may be determined by Formulas (9) and (11). Solving Eq. (12) we get

$$\bar{f}_m^{IV} = L_m^{-1} G_3 \bar{F}_m.$$

The vectors of the minor derivatives may be obtained from the following equalities

$$\bar{f}^i = C_i \bar{f}^{IV} \quad (i = 0, 1, 2, 3).$$

Solution of Eq. (12) is not difficult for any kind of load.

We consider two particular cases.

a) The plate is subjected to a constant load  $q(x, y) = \text{const}$ . Representing load  $q(x, y)$  in the form of (3), we get

$$F_m = \frac{4q}{\pi m}.$$

Whence the vector

$$\bar{F}_m = \frac{4q}{\pi m} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Let the plate be freely supported along edges  $y=0$  and  $y=b$ , then to determine  $f_0^{(4)}$  in Eqs. (11) it should be assumed

$$K_1 = K_3 = 0; \quad K_2 = K_4 = \infty.$$

We consider first a constant-thickness square plate. When the width of the plate is divided into 10 parts it suffices to retain 3 terms of Series (2) to obtain the midplate deflection when  $\mu = 0.3$ :

$$w = 0.0443q \frac{a^4}{Eh^3},$$

which is in complete agreement with the known value. The deflection curve along  $x = \frac{a}{2}$  for a freely supported square plate with a wedge-shaped cross section (Fig. 2a) with  $\frac{h_2}{h_0} = 0.25$  is shown in Fig. 4.

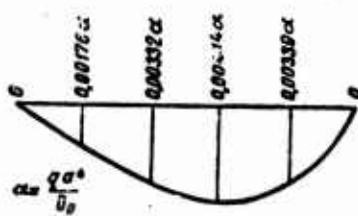


Fig. 4

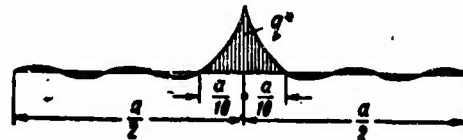


Fig. 5

b) The plate is loaded by a concentrated force  $P$ .

We represent the load in the form

$$q(x, y) = q(x)q(y),$$

where  $q(x)$  and  $q(y)$  are loads per unit length along the  $X$ - and  $Y$ -axes, respectively. In the case of a concentrated force  $q(x)$  may be represented by the series

$$q(x) = \frac{2P}{a} \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \quad m = 1, 2, 3, \dots$$

We shall perform our calculations for a plate divided into 10 width segments, here using the remarks of the preceding section, we consider only one half of the plate, which is broken up into 5 parts. Then the vector of function  $q(y)$  for the case under consideration can be written as

$$q = q^* \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (14)$$

on the condition that we are considering the left-hand side of the cut plate. Since in the method used in these calculations the function is interpolated by Lagrange polynomials, then the load corresponding to Vector (14) has the form shown in Fig. 5. Having reference to the fact that the weight of load  $q(y)$  under consideration should be unity, we get

$$2\omega q^* = 1$$

or

$$q^* = \frac{1}{2\omega}.$$

where  $2\omega$  is the area of the part crosshatched in Fig. 5.

Taking into account the fact that to solve the problem the load should be represented in the form of (3), we have

$$\bar{F}_m = \frac{P}{2\omega a} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Area  $\omega$  is easily determined when the structure of the integral matrix is known. For the case under consideration  $\omega$  will be equal to the difference between the sixth and fifth elements of the extreme right column of matrix  $\Omega$ .

For our case  $\omega = 0,032986 a$ .

Finally we get

$$\bar{F}_m = \frac{15,1579P}{a^3} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (15)$$

Substituting (15) into Formula (12) and retaining 5 terms of Series (2), we get the midplate deflection for  $\mu = 0,3$

$$w = 0,1197 \frac{Pa^3}{Eh^3},$$

which is by 5.3% less than the exact value.

In conclusion we note that using the above method it is not difficult to design a plate for a simultaneous transverse load and compressive forces.

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Transliterated Symbols

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kp = kr = kriticheskiy = critical

## DESIGN OF CONTINUOUS GIRDERS WITH A HELICAL AXIS

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1. We are considering a continuous girder whose axis is a helix, the parametric equation of which has the form (Fig. 1a)<sup>1</sup>

$$x = a \cos t; \quad y = a \sin t; \quad z = \frac{h}{2\pi} t; \quad ds = B dt;$$

$$B = \frac{h}{2\pi} \sqrt{1 + 4\pi^2 \left(\frac{a}{h}\right)^2}. \quad (1)$$

The direction cosines of tangent line  $u$ , of the principal normal  $v$  and of the binormal  $w$  are determined from the respective formulas

$$\cos(u, x) = -\frac{a \sin t}{B}; \quad \cos(u, y) = \frac{a \cos t}{B}; \quad \cos(u, z) = \frac{h}{2\pi B}; \quad (2)$$

$$\cos(v, x) = -\cos t; \quad \cos(v, y) = -\sin t; \quad \cos(v, z) = 0; \quad (3)$$

$$\cos(w, x) = \frac{h}{2\pi B} \sin t; \quad \cos(w, y) = -\frac{h}{2\pi B} \cos t; \quad \cos(w, z) = \frac{a}{B}. \quad (4)$$

The helical girder is supported by stationary ball-and-socket hinges, while the extreme supports are three-dimensional anchors. The girder is subjected to an arbitrary force acting in an arbitrary direction.

Two- and three-term equations of the method of translations in matrix form are written as

$$\begin{aligned} r_{11}\Phi_1 + r_{12}\Phi_2 + R_{1p} &= 0; \\ r_{21}\Phi_1 + r_{22}\Phi_2 + r_{23}\Phi_3 + R_{2p} &= 0; \\ &\dots \dots \dots \\ r_{l-1,l-2}\Phi_{l-2} + r_{l-1,l-1}\Phi_{l-1} + r_{l-1,l}\Phi_l + R_{l-1,p} &= 0; \quad (a) \\ r_{l,l-1}\Phi_{l-1} + r_{l,l}\Phi_l + r_{l,l+1}\Phi_{l+1} + R_{lp} &= 0; \quad (b) \\ &\dots \dots \dots \\ r_{n-2,n-3}\Phi_{n-3} + r_{n-2,n-2}\Phi_{n-2} + r_{n-2,n-1}\Phi_{n-1} + R_{n-2,p} &= 0; \quad (5) \\ r_{n-1,n-2}\Phi_{n-2} + r_{n-1,n-1}\Phi_{n-1} + R_{n-1,p} &= 0, \end{aligned}$$

where:

a) the principal unknowns are the column matrices

$$\Phi_i = \begin{vmatrix} \varphi_x^i \\ \varphi_y^i \\ \varphi_z^i \end{vmatrix}. \quad (6)$$

the elements of which are the angle of rotation  $\varphi_x^i, \varphi_y^i, \varphi_z^i$  of the intermediate support cross sections relative to the three mutually perpendicular axes  $x_i, y_i, z_i$  parallel to the coordinate axes of the helix axis of the contour.

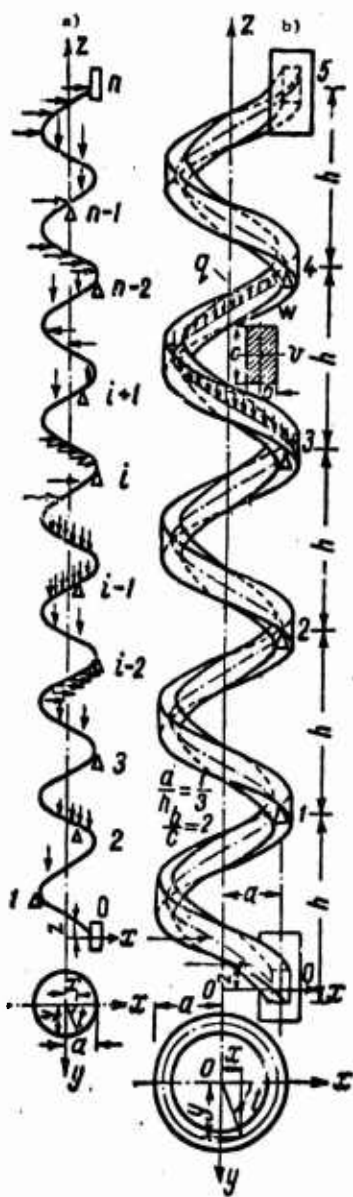


Fig. 1

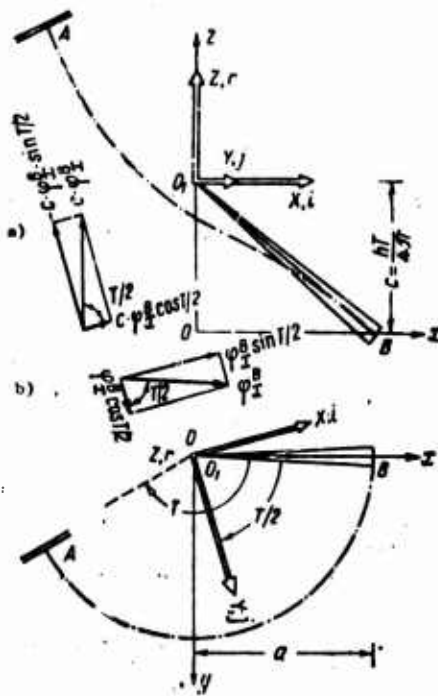


Fig. 2

b) the coefficients of  $\Phi_i$  are the principal and secondary reactions, which are square matrices of the third order of the form

$$r_{ii} = \begin{bmatrix} r_{x_i x_i} & r_{x_i y_i} & r_{x_i z_i} \\ r_{y_i x_i} & r_{y_i y_i} & r_{y_i z_i} \\ r_{z_i x_i} & r_{z_i y_i} & r_{z_i z_i} \end{bmatrix}; \quad r_{i, i+1} = \begin{bmatrix} r_{x_i x_{i+1}} & r_{x_i y_{i+1}} & r_{x_i z_{i+1}} \\ r_{y_i x_{i+1}} & r_{y_i y_{i+1}} & r_{y_i z_{i+1}} \\ r_{z_i x_{i+1}} & r_{z_i y_{i+1}} & r_{z_i z_{i+1}} \end{bmatrix}. \quad (7)$$

The elements of Matrices (7) are unit reactions of the couplings in the principal system, obtained by introducing in the intermediate joints of couplings resisting rotation of the joints relative to axes  $x_i, y_i, z_i$ . For example,  $r_{x_i y_{i+1}}$  is the reaction of the coupling of the  $i$ th joint, preventing this joint from rotating about axis  $x_i$ , this rotation being brought about in the principal system by unit rotation of the  $i + 1$ th joint about axis  $y_{i+1}$ . Matrices  $r_{i, i+1}$  and  $r_{i+1, i}$  are transposed matrices,  $r_{i, i+1} = r_{i+1, i}^T$ .

c) the free terms are the column matrices

$$R_{i,p} = \begin{bmatrix} R_{x_i p} \\ R_{y_i p} \\ R_{z_i p} \end{bmatrix}, \quad (8)$$

the elements of which are reactions in the couplings which were introduced, due to the specified load.

2. To determine the elements of Matrices (7) it is necessary to design a single-level helical girder with ends anchored in space for a load consisting of unit rotations of the top and bottom anchors relative to each of the three mutually perpendicular axes  $x_i, y_i, z_i$ . The design of such girders using the method of forces for certain kinds of force loads, which makes it possible to determine the elements of Matrices (8) is considered in [2]. For designing for rotation the anchors using the principal system used in [2] (Fig. 2) [reactions  $B$  of the bottom anchor have been moved to point  $O_1(0, 0, c = \frac{hT}{4\pi})$ , where  $T$  is a parameter corresponding to top anchor  $A$ ; the principal unknowns  $X, Y, Z$  are mutually perpendicular, with  $X$  and  $Y$  lying in the horizontal plane,  $Y$  forms angle  $\pi/2$  with axis  $+x$  and  $Z$  is directed along axis  $+z$ , the moment vectors  $[M, P]$  are directed, respectively, along  $X, Y, Z$ ], it is first necessary to find the translations in the direction of the principal unknowns, produced by angular translations of the support cross sections.

a) Subjecting bottom anchor  $B$  to a unit angular translation relative to axis  $x_B, \varphi_x^B = 1$  (Fig. 2). The rotation of the anchor produces: 1) linear translation of the end of a perfectly rigid cantilever in the horizontal plane in the negative  $y$  direction, equal to  $\varphi_y^B c$ , the components of which along the  $X, Y, Z$  directions are shown in Fig. 2a; 2) angular translations about axes  $i$  and  $j$ , shown in Fig. 2b.<sup>2</sup>

b) Subjecting bottom anchor  $B$  to a unit angular translation relative to axis  $y_B, \varphi_y^B = 1$  (Fig. 3). This load produces: 1) linear translation of point  $O_1$  in the vertical plane  $xos$ , which is equal to  $O_1B \varphi_y^B$ , and which is broken up onto a component along the  $x$ -axis

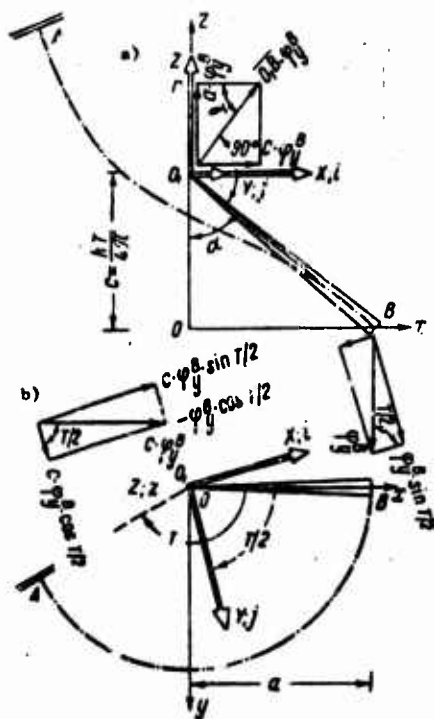


Fig. 3

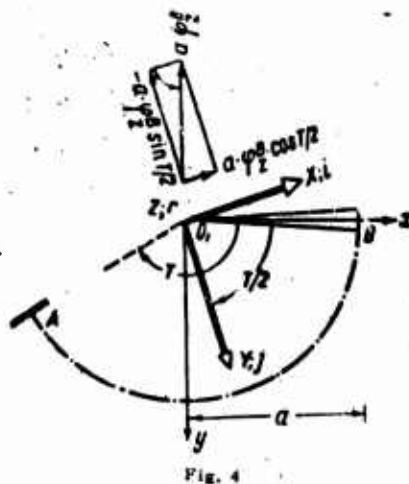
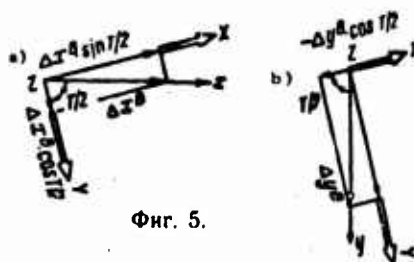


Fig. 4



Фиг. 5.

Fig. 5

equal to  $\overline{O_1 B} \varphi_y^B \sin \alpha = a \varphi_y^B$ , and a horizontal component, parallel to the  $x$ -axis, equal to  $\overline{O_1 B} \varphi_y^B \cos \alpha = c \varphi_y^B$  (Fig. 3a).

The latter supplies translations along the direction of the principal unknowns  $X$  and  $Y$ , the values of which are shown in Fig. 3b; 2) angular translations about axes  $i$  and  $j$ , which are shown in the same figure.

c) Subjecting bottom anchor  $B$  to a unit angular translation relative to axis  $z_B$ ,  $\varphi_z^B = 1$  (Fig. 4). In this case we get: 1) linear translation of point  $O_1$  in the horizontal plane in the negative  $y$  direction, equal to  $a \varphi_z^B$ , which is broken up into components along  $X$  and  $Y$  shown in Fig. 4; 2) angular translation about axis  $r$ , which is unity.

d) Subjecting bottom anchor  $B$  to a unit linear translation along axis  $x_B$ ,  $\Delta x^B = 1$ .

The linear translations of point  $O_1$  are shown in Fig. 5a.

e) Load  $\Delta y^B = 1$ .

The linear translations of point  $O_1$  are shown in Fig. 5b.

f) Load  $\Delta z^B = 1$ .

The linear translation of point  $O_1$  along the  $z$ -axis is unity.

After determining the principal unknowns  $X, Y, Z, i, j, r$  on the basis of [2], the reactions of bottom  $B$  and upper  $A$  anchors relative to axes parallel to the  $x, y, z$  coordinate axes are found from the formulas (Fig. 6):

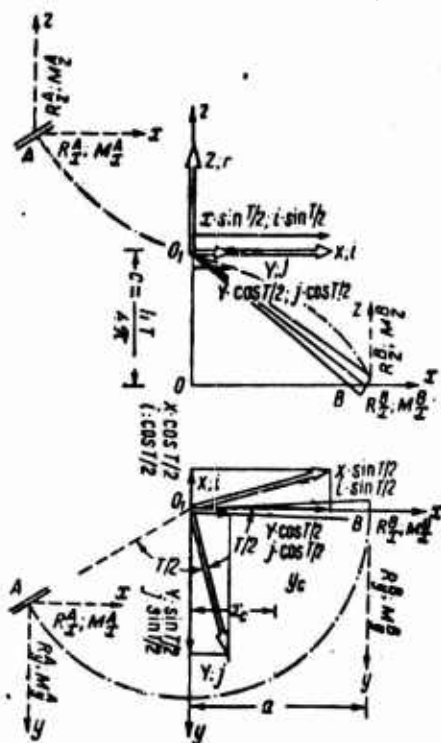


Fig. 6

$$R_x^A = -X \sin \frac{T}{2} - Y \cos \frac{T}{2}; R_y^A = X \cos \frac{T}{2} - Y \sin \frac{T}{2}; R_z^A = -Z; \quad (9)$$

$$M_x^A = X \frac{hT}{4\pi} \cos \frac{T}{2} - Y \frac{hT}{4\pi} \sin \frac{T}{2} + aZ \sin T - i \sin \frac{T}{2} - j \cos \frac{T}{2}; \quad (10)$$

$$M_y^A = X \frac{hT}{4\pi} \sin \frac{T}{2} + Y \frac{hT}{4\pi} \cos \frac{T}{2} - aZ \cos T + i \cos \frac{T}{2} - j \sin \frac{T}{2}; \quad (11)$$

$$M_z^A = -aX \cos \frac{T}{2} - aY \sin \frac{T}{2} - r; \quad (12)$$

$$R_x^B = X \sin \frac{T}{2} + Y \cos \frac{T}{2}; R_y^B = -X \cos \frac{T}{2} + Y \sin \frac{T}{2}; R_z^B = Z; \quad (13)$$

$$M_x^B = X \frac{hT}{4\pi} \cos \frac{T}{2} - Y \frac{hT}{4\pi} \sin \frac{T}{2} + i \sin \frac{T}{2} + j \cos \frac{T}{2}; \quad (14)$$

$$M_y^B = X \frac{hT}{4\pi} \sin \frac{T}{2} + Y \frac{hT}{4\pi} \cos \frac{T}{2} + aZ - i \cos \frac{T}{2} + j \sin \frac{T}{2}; \quad (15)$$

$$M_z^B = aX \cos \frac{T}{2} - aY \sin \frac{T}{2} + r. \quad (16)$$

g) Subjecting top anchor  $A$  to a unit angular translation relative to an axis parallel to the axis  $x, \varphi_x^A = 1$ .

All the reactions of the bottom anchor  $B$  may be determined from the theory of mutual equality of the coupling reactions produced by equal coupling translations:

$$R_{x\varphi_x^A}^B = M_{x\Delta x^B}^A; R_{y\varphi_x^A}^B = M_{x\Delta y^B}^A; R_{z\varphi_x^A}^B = M_{x\Delta z^B}^A; \quad (17)$$

$$M_{x\varphi_x^A}^B = M_{x\varphi_x^B}^A; M_{y\varphi_x^A}^B = M_{x\varphi_y^B}^A; M_{z\varphi_x^A}^B = M_{x\varphi_z^B}^A. \quad (18)$$

The reactions of top anchor  $A$  are determined from equations of statics

$$R_{x\varphi_x^A}^A = -R_{x\varphi_x^A}^B; R_{y\varphi_x^A}^A = -R_{y\varphi_x^A}^B; R_{z\varphi_x^A}^A = -R_{z\varphi_x^A}^B; \quad (19)$$

$$M_{x\varphi_x^A}^A = -M_{x\varphi_x^A}^B - R_{y\varphi_x^A}^B \frac{hT}{2\pi} + R_{z\varphi_x^A}^B a \sin T; \quad (20)$$

$$M_{y\varphi_x^A}^A = -M_{y\varphi_x^A}^B + R_{x\varphi_x^A}^B \frac{hT}{2\pi} + R_{z\varphi_x^A}^B a (1 - \cos T); \quad (21)$$

$$M_{z\varphi_x}^A = -M_{z\varphi_x}^B - R_{x\varphi_x}^B a \sin T - R_{y\varphi_x}^B a (1 - \cos T). \quad (22)$$

h) Load  $\varphi_y^A = 1$ .

All the reactions are determined by analogy with the preceding

$$R_{x\varphi_y}^B = M_{y\Delta x}^A; \quad R_{y\varphi_y}^B = M_{y\Delta y}^A; \quad R_{z\varphi_y}^B = M_{y\Delta z}^A; \quad (23)$$

$$M_{x\varphi_y}^B = M_{y\varphi_x}^A; \quad M_{y\varphi_y}^B = M_{y\varphi_y}^A; \quad M_{z\varphi_y}^B = M_{y\varphi_z}^A; \quad (24)$$

$$R_{x\varphi_y}^A = -R_{x\varphi_y}^B; \quad R_{y\varphi_y}^A = -R_{y\varphi_y}^B; \quad R_{z\varphi_y}^A = -R_{z\varphi_y}^B; \quad (25)$$

$$M_{x\varphi_y}^A = -M_{x\varphi_y}^B - R_{y\varphi_y}^B \frac{hT}{2\pi} + R_{z\varphi_y}^B a \sin T; \quad (26)$$

$$M_{y\varphi_y}^A = -M_{y\varphi_y}^B + R_{x\varphi_y}^B \frac{hT}{2\pi} + R_{z\varphi_y}^B a (1 - \cos T); \quad (27)$$

$$M_{z\varphi_y}^A = -M_{z\varphi_y}^B - R_{x\varphi_y}^B a \sin T - R_{y\varphi_y}^B a (1 - \cos T). \quad (28)$$

i) Load  $\varphi_z^A = 1$ .

$$R_{x\varphi_z}^B = M_{z\Delta x}^A; \quad R_{y\varphi_z}^B = M_{z\Delta y}^A; \quad R_{z\varphi_z}^B = M_{z\Delta z}^A; \quad (29)$$

$$M_{x\varphi_z}^B = M_{z\varphi_x}^A; \quad M_{y\varphi_z}^B = M_{z\varphi_y}^A; \quad M_{z\varphi_z}^B = M_{z\varphi_z}^A; \quad (30)$$

$$R_{x\varphi_z}^A = -R_{x\varphi_z}^B; \quad R_{y\varphi_z}^A = -R_{y\varphi_z}^B; \quad R_{z\varphi_z}^A = -R_{z\varphi_z}^B; \quad (31)$$

$$M_{x\varphi_z}^A = -M_{x\varphi_z}^B - R_{y\varphi_z}^B \frac{hT}{2\pi} + R_{z\varphi_z}^B a \sin T; \quad (32)$$

$$M_{y\varphi_z}^A = -M_{y\varphi_z}^B + R_{x\varphi_z}^B \frac{hT}{2\pi} + R_{z\varphi_z}^B a (1 - \cos T); \quad (33)$$

$$M_{z\varphi_z}^A = -M_{z\varphi_z}^B - R_{x\varphi_z}^B a \sin T - R_{y\varphi_z}^B a (1 - \cos T). \quad (34)$$

The elements of Matrices (7) are found on the basis of (10)-(12), (14)-(16), (18), (20)-(22), (24), (26)-(28), (30), (32)-(34).

3. It follows from the three-term structure of Eqs. (5) that in levels of a continuous helical contour sequentially free of loads the ratio (6) of column matrices is a constant quantity, equal to the angular focus matrix ratios. Thus, if levels 0-1-2 are not loaded (Fig. 1a), then it follows from the first equation of System (5) for  $R_{1p}=0$  that

$$\Phi_1 = k_{21}\Phi_2; \quad k_{21} = -r_{11}^{-1} r_{12}. \quad (35)$$

Precisely in the same manner when levels 0-1-2-3 are free of loads we find from the second equation of the same system with  $R_{1p}=R_{2p}=0$

$$\Phi_2 = k_{32}\Phi_3; \quad k_{32} = -(r_{21}k_{21} + r_{22})^{-1} r_{23}. \quad (36)$$

Finally, if the level sequence 0-1-...-(i-1), i does not carry a load, then when  $R_{1p}=R_{2p}=\dots=R_{i-1,p}=0$  we get

$$\Phi_{i-1} = k_{i,i-1} \Phi_i; \quad k_{i,i-1} = -(r_{i-1,i-2} k_{i-1,i-2} + r_{i-1,i-1})^{-1} r_{i-1,i}. \quad (37)$$

Similarly, from the last of equations of System (5) with levels  $n-(n-1)-(n-2)$  free of load and with  $R_{n-1,p}=0$  we get

$$\Phi_{n-1} = k_{n-2,n-1} \Phi_{n-2}; \quad k_{n-2,n-1} = -r_{n-1,n-1}^{-1} r_{n-1,n-2}. \quad (38)$$

From the equation before the last of the same system in the case of levels  $n-(n-1)-(n-2)-(n-3)$  free of load ( $R_{n-1,p}=R_{n-2,p}=0$ ), we find

$$\begin{aligned} \Phi_{n-2} &= k_{n-3,n-2} \Phi_{n-3}; \\ k_{n-3,n-2} &= -(r_{n-2,n-2} + r_{n-2,n-1} k_{n-2,n-1})^{-1} r_{n-2,n-3}. \end{aligned} \quad (39)$$

Finally, with levels  $n-(n-1)-(n-2)-\dots-(i+1)-i$  not carrying a load, when  $R_{n-1,p}=R_{n-2,p}=\dots=R_{i+1,p}=0$ , we get

$$\Phi_i = k_{i-1,i} \Phi_{i-1}; \quad k_{i-1,i} = -(r_{i,i} + r_{i,i+1} k_{i,i+1})^{-1} r_{i-1,i}. \quad (40)$$

4. To determine the column matrices  $\Phi_{i-1}$  and  $\Phi_i$  when only level  $(i-1)-i$  is carrying a load,  $\Phi_{i-2}$  in Eqs. "a" and "b" of System (5) is expressed in terms of  $\Phi_{i-1}$  from (37), while  $\Phi_{i+1}$  is expressed in terms of  $\Phi_i$  from (39), after which the equations take on the form

$$(r_{i-1,i-2} k_{i-1,i-2} + r_{i-1,i-1}) \Phi_{i-1} + r_{i-1,i} \Phi_i + R_{i-1,p} = 0; \quad (41)$$

$$r_{i,i-1} \Phi_{i-1} + (r_{i,i} + r_{i,i+1} k_{i,i+1}) \Phi_i + R_{i,p} = 0. \quad (42)$$

Solving the system thus obtained, we find:

$$\Phi_i = (k_{i-1,i} k_{i,i+1} - 1)^{-1} [k_{i-1,i} (r_{i-1,i-2} k_{i-1,i-2} + r_{i-1,i-1})^{-1} R_{i-1,p} + (r_{i,i+1} k_{i,i+1} + r_{i,i})^{-1} R_{i,p}]; \quad (43)$$

$$\Phi_{i-1} = (k_{i,i-1} k_{i-1,i} - 1)^{-1} [k_{i,i-1} (r_{i,i+1} k_{i,i+1} + r_{i,i})^{-1} R_{i,p} + (r_{i-1,i-2} k_{i-1,i-2} + r_{i-1,i-1})^{-1} R_{i-1,p}]. \quad (44)$$

The column matrices of the angle of rotation of the remaining joints, located to the bottom and top of the level under load are determined by means of angular focus matrix ratios (37) and (40).

5. The reaction moments in the joints relative to axes  $(x_i, y_i, z_i)$  are obtained from the formulas

$$\begin{aligned} M_x^{i,i+1} &= \bar{M}_{x\varphi_x^i}^{i,i+1} \cdot \varphi_x^i + \bar{M}_{x\varphi_y^i}^{i,i+1} \cdot \varphi_y^i + \bar{M}_{x\varphi_z^i}^{i,i+1} \cdot \varphi_z^i + \bar{M}_{x\varphi_x^{i+1}}^{i,i+1} \cdot \varphi_x^{i+1} + \\ &+ \bar{M}_{x\varphi_y^{i+1}}^{i,i+1} \cdot \varphi_y^{i+1} + \bar{M}_{x\varphi_z^{i+1}}^{i,i+1} \cdot \varphi_z^{i+1} + M_{xp}^{i,i+1}, \end{aligned} \quad (45)$$

$$\begin{aligned} M_y^{i,i+1} &= \bar{M}_{y\varphi_x^i}^{i,i+1} \cdot \varphi_x^i + \bar{M}_{y\varphi_y^i}^{i,i+1} \cdot \varphi_y^i + \bar{M}_{y\varphi_z^i}^{i,i+1} \cdot \varphi_z^i + \bar{M}_{y\varphi_x^{i+1}}^{i,i+1} \cdot \varphi_x^{i+1} + \\ &+ \bar{M}_{y\varphi_y^{i+1}}^{i,i+1} \cdot \varphi_y^{i+1} + \bar{M}_{y\varphi_z^{i+1}}^{i,i+1} \cdot \varphi_z^{i+1} + M_{yp}^{i,i+1}; \end{aligned} \quad (46)$$

$$\begin{aligned} M_z^{i,i+1} &= \bar{M}_{z\varphi_x^i}^{i,i+1} \cdot \varphi_x^i + \bar{M}_{z\varphi_y^i}^{i,i+1} \cdot \varphi_y^i + \bar{M}_{z\varphi_z^i}^{i,i+1} \cdot \varphi_z^i + \bar{M}_{z\varphi_x^{i+1}}^{i,i+1} \cdot \varphi_x^{i+1} + \\ &+ \bar{M}_{z\varphi_y^{i+1}}^{i,i+1} \cdot \varphi_y^{i+1} + \bar{M}_{z\varphi_z^{i+1}}^{i,i+1} \cdot \varphi_z^{i+1} + M_{zp}^{i,i+1}, \end{aligned} \quad (47)$$

in which  $\bar{M}$  denotes the moment about one of the axes, produced by unit rotation of the joint in the principal system, while  $M_{x_p}$ ,  $M_{y_p}$ ,  $M_{z_p}$  are moments due to the specified load under conditions of the principal system.

6. The reactive twisting and bending moments in the joints are found from the formulas

$$M_u^{i,i+1} = M_x^{i,i+1} \cos(x_i u_i) + M_y^{i,i+1} \cos(y_i u_i) + M_z^{i,i+1} \cos(z_i u_i); \quad (48)$$

$$M_v^{i,i+1} = M_x^{i,i+1} \cos(x_i v_i) + M_y^{i,i+1} \cos(y_i v_i) + M_z^{i,i+1} \cos(z_i v_i); \quad (49)$$

$$M_w^{i,i+1} = M_x^{i,i+1} \cos(x_i w_i) + M_y^{i,i+1} \cos(y_i w_i) + M_z^{i,i+1} \cos(z_i w_i). \quad (50)$$

7. We now determine the twisting and bending moments in the joints of a five-level continuous helical girder with its fourth level subjected to a vertical load  $q$  uniformly distributed along the axis of the girder (Fig. 1b).

The single-level helical girder with ends anchored in space is aligned following [2]

$$\begin{aligned} \text{a) } T = 2\pi; a/h = 1/3; c/b = 2; k_1 = EJ_w/C_y = Eb^3c/12Gah^3c = 0,90975 \quad i_{\alpha} \\ G = 0,4E \text{ and } \alpha = 0,229; k_2 = EJ_w/EJ_v = 12cb^3/12bc^3 = 0,25; k_3 = 4\pi^2(a/h)^2 = \\ = 4,38648; B = 0,36938h; i = EJ_w/h. \end{aligned}$$

b) For the principal system shown in Fig. 2 the principal unknowns are determined from the formulas

$$\begin{Bmatrix} X \\ Z \end{Bmatrix} = (\delta_{22}^{-1} \delta_{21} - \delta_{12}^{-1} \delta_{11})^{-1} (\delta_{12}^{-1} \Delta_{1p} - \delta_{22}^{-1} \Delta_{2p}); \quad (51)$$

$$\begin{Bmatrix} i \\ r \end{Bmatrix} = (\delta_{21}^{-1} \delta_{22} - \delta_{11}^{-1} \delta_{12})^{-1} (\delta_{11}^{-1} \Delta_{1p} - \delta_{21}^{-1} \Delta_{2p}); \quad (52)$$

$$\begin{Bmatrix} Y \\ j \end{Bmatrix} = -\delta_{33}^{-1} \Delta_{3p}. \quad (53)$$

in which

$$\delta_{11} = \begin{Bmatrix} \delta_{xx} & \delta_{xz} \\ \delta_{zx} & \delta_{zz} \end{Bmatrix} = \begin{Bmatrix} 0,23277h^3/i & 0,11408h^3/i \\ 0,11408h^3/i & 0,23892h^3/i \end{Bmatrix} \quad (54)$$

$$\delta_{12} = \begin{Bmatrix} \delta_{xi} & \delta_{xr} \\ \delta_{xi} & \delta_{xr} \end{Bmatrix} = \begin{Bmatrix} 0,076045h/i & -0,56706h/i \\ 0 & -0,027148h/i \end{Bmatrix} \quad (55)$$

$$\delta_{21} = \begin{Bmatrix} \delta_{ix} & \delta_{iz} \\ \delta_{rx} & \delta_{rz} \end{Bmatrix} = \begin{Bmatrix} 0,076045h/i & 0 \\ -0,56706h/i & -0,27148h/i \end{Bmatrix} \quad (56)$$

$$\delta_{22} = \begin{Bmatrix} \delta_{ii} & \delta_{ir} \\ \delta_{ri} & \delta_{rr} \end{Bmatrix} = \begin{Bmatrix} 1,36526i & 0 \\ 0 & 2,28199i \end{Bmatrix} \quad (57)$$

$$\delta_{33} = \begin{Bmatrix} \delta_{yy} & \delta_{yj} \\ \delta_{yy} & \delta_{yj} \end{Bmatrix} = \begin{Bmatrix} 0,24833h^3/i & -0,014610h/i \\ -0,014610h/i & 1,36526i \end{Bmatrix} \quad (58)$$

$$\Delta_{1p} = \begin{Bmatrix} \Delta_{xp} \\ \Delta_{zp} \end{Bmatrix}; \Delta_{2p} = \begin{Bmatrix} \Delta_{ip} \\ \Delta_{rp} \end{Bmatrix}; \Delta_{3p} = \begin{Bmatrix} \Delta_{yp} \\ \Delta_{jp} \end{Bmatrix} \quad (59)$$

c) Table 1 gives the free terms of (59) for loads of 12 from "a" to "f" and also for load  $q$ .

TABLE 1

	$\varphi_x^B$	$\varphi_y^B$	$\varphi_z^B$	$\Delta x^B$	$\Delta y^B$	$\Delta z^B$	$q$
$\Delta_{ip}$	$-0,5h$ 0	0 $h/3$	$-h/3$ 0	0 0	1 0	0 1	$-0,13238 h^3 q/l$ $-0,27725 h^3 q/l$
$\Delta_{ap}$	0 0	1 0	0 1	0 0	0 0	0 0	0 $0,031503 h^3 q/l$
$\Delta_{sp}$	0 -1	$-0,5h$ 0	0 0	-1 0	0 0	0 0	$-0,11626 h^3 q/l$ $0,43286 h^3 q/l$

d) Table 2 presents the values of the principal unknowns, calculated from (51)-(53) for the same loads.

e) The reactive moments in the space anchors of a single-level helical girder relative to axes parallel to axes  $x, y, z$ , calculated from the data of ¶2 are given in Table 3.

For load  $q$  it was taken into account that the line of action of the resultant, which is equal to

$$q \int_0^T ds = q2\pi B = 2,32088 hq,$$

is defined by the coordinates

$$x_c = \frac{a \sin T}{T} = 0; \quad y_c = \frac{a(1 - \cos T)}{T} = 0.$$

f) From the data of Table 3 we determine the principal and secondary matrix reactions (7):

$$r_{i,i} = \begin{vmatrix} 14,17616i & 0 & 0 \\ 0 & 6,56116i & 0,97630i \\ 0 & 0,97630i & 1,29118i \end{vmatrix}$$

$$r_{i,i+1} = \begin{vmatrix} 5,62226i & -2,59235i & -1,14639i \\ 2,59235i & -1,26591i & -0,48815i \\ 1,14639i & -0,48815i & -0,64559i \end{vmatrix}$$

$$r_{i,i+1} = \begin{vmatrix} 5,62226i & -2,59235i & -1,14639i \\ -2,59235i & -1,26591i & -0,48815i \\ -1,14639i & -0,48815i & -0,64559i \end{vmatrix}$$

g) The angular focus ratios are found from (37) and (40):

$$k_{21} = \begin{vmatrix} -0,39660; & 0,18287; & 0,080867 \\ -0,29632; & 0,15400 & 0 \\ -0,66380; & 0,26161 & 0,50000 \end{vmatrix}$$

$$k_{32} = \begin{vmatrix} -0,43268; & 0,20263; & 0,040920 \\ -0,26156; & 0,13734; & 0,010506 \\ -0,64515; & 0,23764; & 0,68295 \end{vmatrix}$$

TABLE 2

	$\varphi_x^B$	$\varphi_y^B$	$\varphi_z^B$	$\Delta_x^B$	$\Delta_y^B$	$\Delta_z^B$	$q$
X	-12,71031i/h	-5,22778i/h	-2,29278i/h	0	25,42067i/h <sup>2</sup>	-11,43567i/h <sup>2</sup>	0
Y	-0,043119	-2,01467	0	-4,02934i/h <sup>2</sup>	0	0	0,4979hq
Z	5,71783	3,74881	1,08131	0	-11,43566	9,33556	-1,16043
i	0,70797i	1,02365i	0,12771i	0	-1,41596i/h	0,13698i/h	0
j	-0,73290	-0,021560	0	-0,043119i/h	0	0	-0,31223h <sup>2</sup> q
r	-3,09042	-1,25449	-0,11867	0	6,18084	-2,73064	0

TABLE 3

	$M_x^A$	$M_y^A$	$M_z^A$	$M_x^B$	$M_y^B$	$M_z^B$
$\varphi_x^A$	7,08808i	-2,63545i	-1,14639i	5,62226i	2,59235i	1,14639i
$\varphi_y^A$	-2,63545i	3,28058i	0,48815i	-2,59235i	-1,26591i	-0,48815i
$\varphi_z^A$	-1,14639i	0,48815i	0,64559i	-1,14639i	-0,48815i	-0,64559i
$\varphi_x^B$	5,62226i	-2,59235i	-1,14639i	7,08808i	2,63545i	1,14639i
$\varphi_y^B$	2,59235i	-1,26591i	-0,48815i	2,63545i	3,28058i	0,48815i
$\varphi_z^B$	1,14639i	-0,48815i	-0,64559i	1,14639i	0,48815i	0,64559i
q	-0,31223h <sup>2</sup> q	0,16192h <sup>2</sup> q	0	0,31223h <sup>2</sup> q	0,16192h <sup>2</sup> q	0

$$k_{43} = \begin{vmatrix} -0,42946; & 0,20181; & 0,033379 \\ -0,25438; & 0,13452; & 0,0052770 \\ -0,70969; & 0,26034; & 0,76183 \end{vmatrix}$$

$$k_{34} = \begin{vmatrix} -0,39660; & -0,18287; & -0,080867 \\ 0,29632; & 0,15400; & 0 \\ 0,66380; & 0,26161; & 0,50000 \end{vmatrix}$$

h) The principal unknowns of a continuous helical girder are found from (43), (44), (36) and (35):

$$\Phi_4 = (k_{34}k_{43} - 1)^{-1} [k_{34}(r_{32}k_{32} + r_{33})^{-1}R_{3p} + r_{44}^{-1}R_{4p}] =$$

$$= \begin{vmatrix} 0,064300qh^2/i \\ -0,056912qh^2/i \\ -0,067603qh^2/i \end{vmatrix}$$

$$\Phi_3 = (k_{43}k_{34} - 1)^{-1} [k_{43}r_{44}^{-1}R_{4p} + (r_{32}k_{32} + r_{33})^{-1}R_{3p}] =$$

$$= \begin{vmatrix} -0,073312qh^2/i \\ -0,047919qh^2/i \\ -0,054859qh^2/i \end{vmatrix}$$

$$\Phi_2 = k_{33}\Phi_3 = \begin{vmatrix} 0,019766qh^2/i \\ 0,012019qh^2/i \\ -0,0015544qh^2/i \end{vmatrix}; \quad \Phi_1 = k_{21}\Phi_2 = \begin{vmatrix} -0,0057671qh^2/i \\ -0,0040063qh^2/i \\ -0,010753qh^2/i \end{vmatrix}$$

i) The angular reactive moments about axes  $x_i, y_i, z_i$  are determined from (45)-(47):

$$\begin{aligned} M_x^{01} &= -0,0097110h^2q & M_x^{10} &= -M_x^{12} = -0,017993h^2q \\ M_y^{01} &= -0,0046293 & M_y^{10} &= -M_y^{12} = -0,0031931 \\ M_z^{01} &= 0,0022864 & M_z^{10} &= -M_z^{12} = -0,0022864 \\ M_x^{21} &= -M_x^{23} = 0,055070h^2q & M_x^{32} &= -M_x^{34} = -0,18993h^2q \\ M_y^{21} &= -M_y^{23} = 0,011849 & M_y^{32} &= -M_y^{34} = -0,056465 \\ M_z^{21} &= -M_z^{23} = -0,0022874 & M_z^{32} &= -M_z^{34} = -0,0022876 \\ M_x^{43} &= -M_x^{45} = -0,22827h^2q & M_x^{54} &= 0,13647h^2q \\ M_y^{43} &= -M_y^{45} = 0,050250 & M_y^{54} &= -0,061645 \\ M_z^{43} &= -M_z^{45} = -0,0022870 & M_z^{54} &= -0,0022870 \end{aligned}$$

k) The angular reactive twisting and bending moments are found from (48)-(50):

$$\begin{aligned} M_u^{01} &= -0,0031925h^2q & M_u^{10} &= -M_u^{12} = -0,0038666h^2q \\ M_v^{01} &= 0,0097110 & M_v^{10} &= -M_v^{12} = 0,017993 \\ M_w^{01} &= 0,0040579 & M_w^{10} &= -M_w^{12} = -0,00068750 \\ M_u^{21} &= -M_u^{23} = 0,0097074h^2q & M_u^{32} &= -M_u^{34} = -0,051941h^2q \\ M_v^{21} &= -M_v^{23} = -0,055070 & M_v^{32} &= -M_v^{34} = 0,18995 \\ M_w^{21} &= -M_w^{23} = -0,0071696 & M_w^{32} &= -M_w^{34} = 0,22265 \\ M_u^{43} &= -M_u^{45} = 0,044362h^2q & M_u^{54} &= -0,056615h^2q \\ M_v^{43} &= -M_v^{45} = 0,22827 & M_v^{54} &= -0,13647 \\ M_w^{43} &= -M_w^{45} = 0,023715 & M_w^{54} &= 0,024497 \end{aligned}$$

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## Footnotes

- |     |   |
|-----|---|
| 263 | <sup>1</sup> Helical girders are encountered in reinforced-concrete structures of staircases, of reinforced-concrete girders of garage ramps, etc.      |
| 265 | <sup>2</sup> In [2] these translations are denoted by $\Delta_{xy}$ , $\Delta_{yz}$ , $\Delta_{xz}$ and $\Delta_{yx}$ , $\Delta_{zy}$ , $\Delta_{zx}$ . |
| 266 | <sup>3</sup> In designing a continuous contour for linear settling of supports resort is also had to data of "d," "e" and "f."                          |

**Engineer K. Boyadzhiev**

The solution of certain statically indeterminate systems (continuous beams, unbraced trusses, single-story frames, etc.) involves consideration of systems of three-term equations:

In many cases it is necessary to represent the unknowns as a linear function in the form

where  $A_1, A_2, \dots, A_n$  are the free terms of Eq. (1a), while  $\lambda_{ik}$  are elements of the inverse matrix.

The matrix composed of the coefficients of an equation is denoted by  $A$ , while its inverse is denoted by  $A^{-1}$ :

A number of studies has just been published during the last several years, which involve finding the elements of the inverse matrix of three-term equations. It could have been expected that each successive method would yield a better solution and would have some advantage over the preceding methods. However, some of the latest methods are not better solutions but, conversely, include a greater number of arithmetic operations than the preceding method. Incorrect comparison of certain methods is permitted.

In the present article we consider some of the lately published methods and we show their interrelationship and shortcomings.

For this purpose we define  $\lambda_{ik}$  by eliminating unknowns from canonical equations ( $a_{ik} \neq a_{ki}$ ).

Let there be given system of equations (1a).

Starting the elimination with the first equation, we can get

$$X_{k-1} = a'_{k-1} X_k + r'_{k-1,1} A_1 + r'_{k-1,2} A_2 + \dots + r'_{k-1,k-1} A_{k-1}.$$

Substituting into the  $k$ th equation, we will have

$$X_k = a'_k X_{k+1} + r'_{k1} A_1 + r'_{k2} A_2 + \dots + r'_{kk} A_k; \quad (1d)$$

$$\left. \begin{aligned} \text{for } a'_k &= \frac{a_{k,k-1}}{\gamma_k} \quad \gamma_k = a_{kk} + a_{k,k-1} a'_{k-1}; \quad r'_{k1} = \frac{r_{k-1,1} a_{k,k-1}}{\gamma_k}; \\ r_{k,k-1} &= \frac{r_{k-1,k-1} a_{k,k-1}}{\gamma_k} \quad r'_{kk} = -\frac{1}{\gamma_k}. \end{aligned} \right\} \quad (1e)$$

In the same manner, starting elimination from the last equation, we get

$$X_{k+1} = a'_{k+1} X_k + r'_{k+1,k+1} A_{k+1} + \dots + r'_{k+1,n} A_n, \quad (1f)$$

where the coefficients are analogous to (1e).

Solving simultaneously (1d) and (1f) and introducing the notation

$$(1 - a'_k a'_{k+1}) = \varphi'_k,$$

we have

$$X_k = \frac{1}{\varphi'_k} [r'_{k1} A_1 + r'_{k2} A_2 + \dots + r'_{kk} A_k + a'_{k+1} (r'_{k+1,k+1} A_{k+1} + \dots + r'_{k+1,n} A_n)]. \quad (1g)$$

Consequently:

$$\left. \begin{aligned} \lambda_{k1} &= \frac{r'_{k1}}{\varphi'_k} = \frac{r'_{k-1,1} a_{k,k-1}}{\gamma_k \varphi'_k} = -\frac{\lambda_{k-1,1} a_{k,k-1} \varphi'_{k-1}}{\gamma_k \varphi'_k}; \\ \lambda_{ks} &= \frac{r'_{k-1,s}}{\varphi'_k} = \frac{r'_{k-1,s} a_{k,k-1}}{\gamma_k \varphi'_k} = -\frac{\lambda_{k-1,s} a_{k-1,k} \varphi'_{k-1}}{\gamma_k \varphi'_k}; \\ \lambda_{kk} &= \frac{r'_{kk}}{\varphi'_k} = \frac{1}{\gamma_k \varphi'_k}. \end{aligned} \right\} \quad (1h)$$

If, having resort to Eqs. (1d) and (1f), we express  $X_{k+1}$  in terms of  $X_k$ , then we get the following relationships:

$$\varphi'_{k+1} = 1 - a'_k a'_{k+1} = \varphi'_k; \quad \lambda_{kk} = \frac{r'_{kk}}{\varphi'_k} = \frac{1}{\gamma_k \varphi'_k} = \frac{1}{\gamma_k \varphi_k}. \quad (1i)$$

It follows from Eq. (1i) that  $\frac{\varphi'_{k-1}}{\varphi'_k \gamma_k} = \frac{1}{\gamma_k}$  and Formulas (1h) take on the form

$$\left. \begin{aligned} \lambda_{ks} &= -\frac{\lambda_{k-1,s} a_{k-1,k}}{\gamma_k} \quad \text{for } 1 \leq s \leq k-1; \\ \lambda_{ks} &= -\frac{\lambda_{k-1,s} a_{k-1,k}}{\gamma_k} \quad \text{for } k+1 \leq s \leq n; \\ \lambda_{kk} &= \frac{1}{\varphi_k \gamma_k} = \frac{1}{\varphi_k \gamma_k}. \end{aligned} \right\} \quad (1k)$$

After elementary transformations we can write

$$\begin{aligned} \gamma_k' &= a_{kk} + a_{k-1}' a_{k-1,k} = a_{kk} - \frac{a_{k-1,k} a_{k,k-1}}{\gamma_{k-1}}; \\ \gamma_k' &= a_{kk} + a_{k-1}' a_{k+1,k} = a_{kk} - \frac{a_{k+1,k} a_{k,k+1}}{\gamma_{k+1}}; \\ \varphi_k \gamma_k' &= \varphi_k' \gamma_k' = \gamma_k' (1 - a_{k-1}' a_{k+1,k}) = \gamma_k' \left( 1 - \frac{a_{k,k+1} a_{k+1,k}}{\gamma_k \gamma_{k+1}} \right) = \\ &= \gamma_k' - \frac{a_{k,k+1} a_{k+1,k}}{\gamma_{k+1}} = \gamma_k' + \gamma_k' - a_{kk}. \end{aligned} \quad (1l)$$

We assume that

$$\omega_k = \gamma_k' + \gamma_k' - a_{kk}. \quad (1m)$$

Thus, the solution is performed in the following sequence:

- 1)  $\gamma_k', \gamma_k''$  are determined from Formulas (1l);
- 2)  $\omega_k$  is determined from Formula (1m);
- 3)  $\lambda_{kk}$  is determined from the formula  $\lambda_{kk} = \frac{1}{\omega_k}$ ;
- 4)  $\lambda_{ks}$  is determined from Formula (1k).

The evaluation of  $\lambda_{ik}$  may be tabulated.

Reference [1] gives the following formulas for  $\lambda_{ik}$ :

$$\left. \begin{aligned} \gamma_1' &= a_{11}; \gamma_2' = a_{22} - \frac{a_{21}^2}{\gamma_1}; \gamma_n' = a_{nn}; \gamma_{n-1}' = a_{nn} - \frac{a_{n,n-1}}{\gamma_n}; \\ \gamma_k' &= a_{kk} - \frac{a_{k,k-1}^2}{\gamma_{k-1}}; \gamma_k'' = a_{kk} - \frac{a_{k,k+1}^2}{\gamma_{k+1}}; \end{aligned} \right\} \quad (2a)$$

$$\begin{aligned} \alpha_1' &= 1; \alpha_2' = \frac{1}{a_{2,1}} \alpha_1' \gamma_1'; \alpha_n' = 1; \\ \alpha_k &= \frac{1}{a_{k,k-1}} \alpha_{k-1}' \gamma_{k-1}'; \alpha_k'' = \frac{1}{a_{k,k+1}} \alpha_{k+1}' \gamma_{k+1}'. \end{aligned}$$

$$\omega_k = a_{kk} - \frac{a_{k,k-1}^2}{\gamma_{k-1}} = \frac{a_{k,k+1}^2}{\gamma_{k+1}}; \quad (2b)$$

$$\left. \begin{aligned} \lambda_{ki} &= \frac{a_i}{a_k \omega_k} (-1)^{k+1} \quad k \geq i; \\ \lambda_{ki} &= \frac{a_i}{a_k \omega_k} (-1)^{k+1} \quad k \leq i; \\ \lambda_{kk} &= \frac{1}{\omega_k}. \end{aligned} \right\} \quad (2c)$$

These formulas are close to Expressions (1k), (1l) and (1m). Using elementary operations, Formulas (2b) may be reduced to (1m). If we determine the ratio of  $\lambda_{ki}$  and  $\lambda_{k-1,i}$  when  $k \geq i$ , then we will get from Formulas (2c)

$$\begin{aligned} -\frac{\lambda_{ki}}{\lambda_{k-1,i}} &= \frac{a_i}{a_k \omega_k} : \frac{a_i}{a_{k-1} \omega_{k-1}} = \frac{a_{k-1} \omega_{k-1}}{a_k \omega_k} = \\ &= \frac{a_{k-1} \omega_{k-1} a_{k,k-1}}{a_{k-1} \omega_k \gamma_{k-1}} = \frac{a_{k,k-1} \omega_{k-1}}{\gamma_{k-1} \omega_k}. \end{aligned}$$

It can be proven that

$$\frac{\omega_{k-1}}{\omega_k \gamma_{k-1}} = \frac{1}{\gamma_k},$$

whence it follows

$$-\frac{a_{ki}}{\lambda_{k-1,i}} = \frac{a_{k,k-1}}{\gamma_k}; \quad -a_{ki} = \frac{a_{k,k-1} \lambda_{k-1,i}}{\gamma_k} \quad \text{when } k \geq i$$

and, accordingly,

$$-\lambda_{ki} = \frac{a_{k,k-1} \lambda_{k-1,i}}{\gamma_k} \quad \text{when } k \leq i,$$

which yields Formula (1k).

It follows from the above transformations that it is possible to calculate  $\lambda_{i,k}$  without resort to coefficients  $a_k$  and  $a_k^*$ . These formulas may also be used for  $a_{k-1,k} \neq a_{k,k-1}$ , it is only necessary to substitute in Formula (2a) instead of  $a_{k-1,k}^2$ , respectively,  $a_{k-1,k} a_{k,k-1}$ .

M.M. Filonenko-Borodich has obtained in his article [2]

$$\left. \begin{aligned} k'_1 &= -\frac{a_{1,1}}{a_{1,2}}; \quad k'_2 = -\frac{1}{a_{2,1}} \left( a_{22} + \frac{a_{2,1}}{k'_1} \right); \\ k'_i &= -\frac{1}{a_{i+1,i}} \left( a_{ii} + \frac{a_{i,i-1}}{k'_{i-1}} \right); \\ k'_n &= -\frac{a_{nn}}{a_{n,n-1}}; \quad k'_{n-1} = -\frac{1}{a_{n-1,n-2}} \left( a_{n-1,n-1} + \frac{a_{n,n-1}}{k'_n} \right) k'_i = \\ &= -\frac{1}{a_{i,i+1}} + \frac{b_{i,i+1}}{k'_{i+1}}. \\ -\omega_k &= \frac{a_{i,i-1}}{k'_{i-1}} + \frac{a_{i,i-1}}{k'_{i+1}} - a_{ii}; \end{aligned} \right\} \quad (3a)$$

$$(3b)$$

$$\begin{aligned}
& \lambda_{ii} = \frac{1}{\omega_i}; \\
& -\lambda_{i-1,i} = \frac{\lambda_{ii}}{k'_{i-1}}; \quad -\lambda_{i+1,i} = \frac{\lambda_{ii}}{k'_{i+1}}; \\
& -\lambda_{i-2,i} = \frac{\lambda_{ii}}{k'_{i-1} k'_{i-2}}; \quad -\lambda_{i+2,i} = \frac{\lambda_{ii}}{k'_{i+1} k'_{i+2}}; \\
& \dots \dots \dots \\
& -\lambda_{i,s} = \frac{\lambda_{ii}}{k'_{i-1} k'_{i-2} \dots k'_2}; \quad -\lambda_{n-1,i} = \frac{\lambda_{ii}}{k'_{i+1} k'_{i+2} \dots k'_{n-1}}; \\
& -\lambda_{i,1} = \frac{\lambda_{ii}}{k'_{i-1} k'_{i-2} \dots k'_2 k'_1}; \quad -\lambda_{ni} = \frac{\lambda_{ii}}{k'_{i+1} k'_{i+2} \dots k'_n k'_{n-1} k'_n}.
\end{aligned} \tag{3c}$$

It is easily noticed that upon denoting

$$-\gamma'_{k-1} = a_{i-1,i} k'_{i-1}, \quad -\gamma'_{k+1} = a_{i+1,i} k'_{i+1}$$

and substituting into Formulas (3a), (3b) and (3c), we get Expressions (1k), (1l) and (1m).

We consider the determination of  $\lambda_{ik}$  using formulas of [3].

For  $a_{h-1,k} = a_{h,k-1}$

$$\begin{aligned}
& c'_{1,2} = -\frac{a_{12}}{a_{11}}; \quad c'_{n,n-1} = -\frac{a_{n,n-1}}{a_{nn}}; \\
& c'_{23} = -\frac{a_{23}}{a_{22} + a_{21} c'_{12}}; \quad c'_{n-1,n-2} = -\frac{a_{n-1,n-2}}{a_{n-1,n-1} + a_{n-1,n} c'_{n,n-1}}; \\
& \dots \dots \dots
\end{aligned} \tag{4a}$$

$$\begin{aligned}
& c'_{i,i+1} = \frac{a_{i,i+1}}{a_{i,i} + a_{i,i-1} c'_{i-1,i}}; \quad c'_{i,i-1} = -\frac{a_{i,i-1}}{a_{i,i} + a_{i,i+1} c'_{i+1,i}}; \\
& \omega_i = a_{i,i-1} c'_{i-1,i} + a_{i,i+1} c'_{i+1,i} + a_{ii}.
\end{aligned} \tag{4b}$$

$$\begin{aligned}
& \lambda_{ii} = \frac{1}{\omega_i}; \\
& \left. \begin{aligned} \lambda_{i+1,i} &= \lambda_{ii} c'_{i+1,i}; & \lambda_{i-1,i} &= \lambda_{ii} c'_{i-1,i}; \\ \lambda_{i+1,s} &= \lambda_{is} c'_{i+1,i}; & \lambda_{i-1,s} &= \lambda_{is} c'_{i-1,i}. \end{aligned} \right\}
\end{aligned} \tag{4c}$$

If we denote

$$-\gamma'_{k-1} = \frac{a_{i-1,i}}{c'_{i-1,i}}, \quad -\gamma'_{k+1} = \frac{a_{i+1,i}}{c'_{i+1,i}},$$

then from the above formulas we again get Expressions (1k), (1l) and (1m).

In [4] formulas for  $\lambda_{ik}$  are found by means of determinants. These are derived as follows.

We calculate  $\Delta_i^v$  and  $\Delta_i^n$ :

$$\begin{aligned}
 \Delta_1^* &= a_{11} & \Delta_1^* &= a_{nn} \\
 \Delta_2^* &= \Delta_1^* a_{22} - a_{12} a_{21} & \Delta_2^* &= \Delta_1^* a_{22-i, n-1} - a_{n, n-1} a_{n-1, n} \\
 &\dots & & \dots \\
 \Delta_i^* &= \Delta_{i-1}^* a_{ii} - a_{i-1, i} a_{i, i-1} \Delta_{i-2}^* \\
 \Delta_n^* &= \Delta_n^* = \Delta_n^*
 \end{aligned} \quad (5a)$$

In this case the inverse matrix for  $n=4$  takes on the form

$$A^{-1} = \frac{1}{\Delta_4} \begin{vmatrix} \Delta_3^* & -a_{12}\Delta_2^* & a_{12}a_{23}\Delta_1^* & -a_{12}a_{23}a_{34} \\ -a_{21}\Delta_2^* & \Delta_2^*\Delta_1^* & -a_{23}\Delta_1^*\Delta_1^* & a_{23}a_{34}\Delta_1^* \\ a_{21}a_{32}\Delta_1^* & -a_{32}\Delta_1^*\Delta_1^* & \Delta_1^*\Delta_2^* & -a_{34}\Delta_2^* \\ -a_{21}a_{32}a_{43} & a_{32}a_{43}\Delta_1^* & -a_{43}\Delta_2^*\Delta_3^* & \end{vmatrix} \quad (5b)$$

It should be noted that this approach is most unlikely to be used in practice since the number of arithmetic operations in it is excessive.

Comparing the formulas for  $\lambda_{ik}$  suggested by different authors we note that it is possible to pass from one set of formulas to another solely by means of simple mathematical transformations; here the method given in [3] should be taken as the principal method for finding  $\lambda_{ik}$ .

Формулы 1	Операции 2	3 Количество операций
(1 к), (1 л), (1 м) 4	5 Сложение 6 Умножение 7 Записи	$3n - 4$ $n^2 + 6n - 6$ $n^2 + 3n - 2$
А. П. Филина и Е. С. Гребень 8	5 Сложение 6 Умножение 7 Записи	$3n - 4$ $n^2 + 10n - 4$ $n^2 + 5n$
М. М. Филонен- ко-Бородича 9	5 Сложение 6 Умножение 7 Записи	$3n - 4$ $n^2 + 6n - 6$ $n^2 + 3n - 2$
А. А. Уманского 10	5 Сложение 6 Умножение 7 Записи	$3n - 4$ $n^2 + 6n - 6$ $n^2 + 3n - 2$
Ю. Г. Пчелкина 11	5 Сложение 6 Умножение 7 Записи	$2n - 2$ $4n^2 - 4n - 4$ $2n^2 - 3n + 4$

- |                         |                                 |
|-------------------------|---------------------------------|
| 1) Formulas             | 7) Results                      |
| 2) Operations           | 8) A.P. Filin and Ye.S. Greben' |
| 3) Number of operations | 9) M.M. Filonenko-Borodich      |
| 4) (1k), (1l), (1m)     | 10) A.A. Umanskiy               |
| 5) Addition             | 11) Yu.G. Pchelkin.             |
| 6) Multiplication       |                                 |

The table presents the number of operations needed for evaluating the coefficients.

It follows from the table that the number of arithmetic operations using Formulas (1), (3) and (4) is the same. The number of multiplications may be reduced by increasing the number of results, but this is not preferable. Formulas (1) have certain advantages, since the multiplications are performed in succession.<sup>1</sup> Reference [1] presents a table where it is pointed out that the total number of arithmetic operations according to the method suggested is 62 and 48%, respectively, for  $n=10$  and  $n=20$  of the total number of arithmetic operations in the M.M. Filonenko-Borodich method. The table presented here gives other relationships between the volume of arithmetic operations under the two methods. Reference [1] incorrectly determined the number of operations in the M.M. Filonenko-Borodich method and incorrectly claims that the method suggested in [1] involves a lesser number of arithmetic operations.

It additionally follows from the table that the Yu.G. Pchelkin method is entirely unsuitable for computations, particularly for a large number of equations.

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#### Footnote

- 281      <sup>1</sup>Expressions  $a + \frac{bc}{d}$  and  $a\left(b + \frac{c}{d}\right)$  involve the same arithmetic operations, but the first of them is more convenient to calculate.

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Transliterated Symbols

279      в = v = vysshly = upper

279      н = n = nishhy = lower

# DESIGN OF A FLEXIBLE STRING FOR SIMULTANEOUSLY APPLIED VERTICAL AND HORIZONTAL LOADS

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## §1. General Remarks

The design of a flexible string subjected simultaneously to vertical and horizontal loads is a substantially more complicated problem than the ordinary problem of the equilibrium of a string subjected to a vertical load only.

The horizontal tension in a string in the presence of a horizontal load varies along the span, which highly complicates all the calculations. In determining the horizontal tension in any cross section (for example, at the left support) and then in finding the translations one has to deal with non-linear equations.

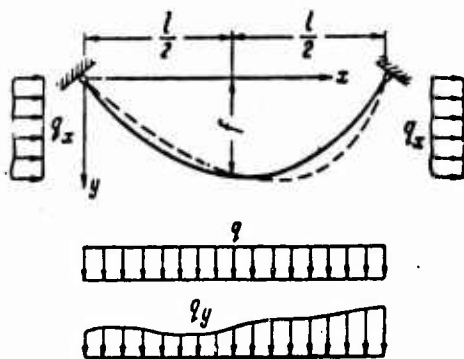


Fig. 1

The present paper uses the method of iterations together with numerical integration of differential equations.

We consider the case when a flexible string is subjected to the following loads.

1. Constant uniformly distributed (along the horizontal projection) load  $g$  t/m.

2. Momentary vertical load  $q_y$  (distributed along the horizontal projection), with an arbitrary distribution.

3. Momentary horizontal load  $q_x$ , acting in the plane of the flexible string (distributed along the vertical projection), with an arbitrary distribution.

We shall assume that under the action of constant load  $g$  the string has taken up a certain position, and its basic dimensions in this case are  $l$  and  $f$ .

After this, loads  $q_y$  and  $q_x$  are applied to the string and due to them it occupies a new position. The problem consists in determining the vertical and horizontal translations of the points on the string's axis and the forces in them. In solving this problem we shall take into account the string's elongations due to these loads. It will be assumed that the stiffness of the string is constant over its length and equal to  $EF$ .

The problem will thus be solved in its general statement, for which it is also possible to solve particular cases, by setting one or another load equal to zero, or by adjusting the string's stiffness.

Figure 1 depicts the flexible string and the diagrams of the above loads. The heavy line shows the outline of the string's axis produced by constant load  $g$ , while the dashed line denotes the sought position which the string will take up upon application of loads  $q_y$  and  $q_x$ .

We shall assume that the supports are situated at the same level and that the constant load is uniformly distributed. In this case the outline of the string will be a quadratic parabola with the equation

$$y = \frac{4fx}{l^2} (l - x).$$

The horizontal tension in the string due to the constant load is defined by the expression

$$H_x = \frac{gl^2}{8f}.$$

## §2. Derivation of Basic Equations

To derive the basic equations we shall consider an element cut out in any section of the deformed string (Fig. 2). The forces acting on this element are defined by the expressions

$$V_y = (g + q_y) dx;$$

$$V_x = q_x |dy|;$$

$$Q(x) = H(x) \frac{d(y + \eta)}{dx};$$

$$Q(x) + dQ(x) = [H(x) + dH(x)] \left[ \frac{d(y + \eta)}{dx} + \frac{d^2(y + \eta)}{dx^2} dx \right].$$

Setting up the equation of projection onto the vertical and horizontal axes, we get

$$\begin{aligned} -H(x) \frac{d(y + \eta)}{dx} + [H(x) + dH(x)] \left[ \frac{d(y + \eta)}{dx} + \frac{d^2(y + \eta)}{dx^2} dx \right] + \\ + (g + q_y) dx = 0; \\ dH(x) + q_x |dy| = 0. \end{aligned}$$

Disregarding products  $dH(x) d\eta$  and  $dH(x) \frac{d^2(y+\eta)}{dx^2} dx$  and transforming, we have

$$H(x) \frac{d^2\eta}{dx^2} + H(x) \frac{d^2y}{dx^2} + \frac{dH(x)}{dx} \cdot \frac{dy}{dx} + (g + q_y) = 0; \quad (1)$$

$$\frac{dH(x)}{dx} = -q_x \left| \frac{dy}{dx} \right|. \quad (2)$$

Substituting Expression (2) into Eq. (1), we get

$$\frac{d^2\eta}{dx^2} = -\frac{d^2y}{dx^2} - \frac{g + q_y}{H(x)} + \frac{q_x}{H(x)} \left| \frac{dy}{dx} \right| \frac{dy}{dx}. \quad (3)$$

To establish the relationship between the horizontal and vertical translations we turn to Fig. 3, which depicts this element in the initial and displaced states. The vertical translations  $\eta$  shall be regarded as positive if they are directed downward, while the horizontal translations  $u$  will be regarded as positive when moving to the right. We set up an expression for the difference of squares of the lengths of segments  $a_1b_1$  and  $ab$ :

$$(ds + \Delta ds)^2 - (ds)^2 = (dx + du)^2 + (dy + d\eta)^2 - (dx)^2 - (dy)^2.$$

Disregarding expressions  $(\Delta ds)^2$ ,  $(du)^2$  and  $(d\eta)^2$ , we have

$$ds\Delta ds = dxdu + dyd\eta. \quad (4)$$

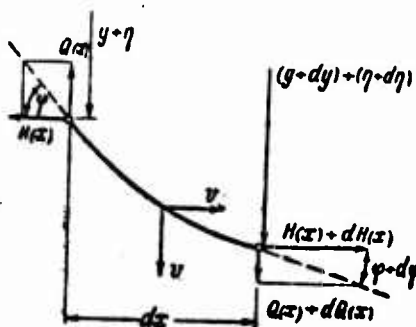


Fig. 2

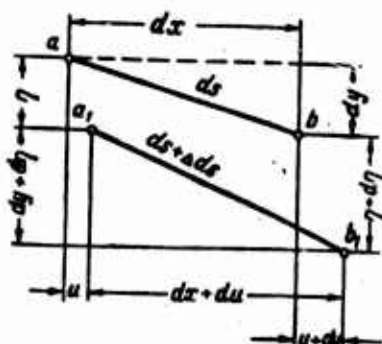


Fig. 3

Quantity  $\Delta ds$  is the elongation of element  $ds$  due to momentary loads  $q_y$  and  $q_x$ :

$$\Delta ds = \frac{H(x) - H_g}{\cos \varphi EF} ds,$$

where  $\varphi$  is the angle made by the tangent with the axis of the flexible string. Substituting this expression into Eq. (4), we get

$$\frac{du}{dx} = -\frac{dy}{dx} \cdot \frac{d\eta}{dx} + \frac{H(x) - H_g}{\cos \varphi EF} \left( \frac{ds}{dx} \right)^2.$$

Integrating and making use of the fact that

$$\frac{1}{\cos \varphi} \left( \frac{ds}{dx} \right)^2 = \frac{1}{\cos^3 \varphi} = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2}$$

and

$$\int \frac{dy}{dx} \cdot \frac{d\eta}{dx} dx = \eta \frac{dy}{dx} + \int \eta \frac{d^2y}{dx^2} dx,$$

we have<sup>2</sup>

$$u = -\frac{dy}{dx} \eta + \int \frac{d^2y}{dx^2} \eta dx + \frac{H(x) - H_g}{EF} \int \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2} dx. \quad (5)$$

We thus have two equations (3) and (5) for determining the translations. However, these equations do not suffice for solving the problem, since in addition to the unknown  $\eta$  and  $u$  it is also necessary to determine horizontal tension  $H(x)$ , which varies along the span. To set up the third equation we use the boundary conditions at  $x=l$ ;  $u=0$ ,  $\eta=0$ , which brings us to the basic equation

$$\int_0^l \frac{H(x) - H_g}{EF} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2} dx + \int_0^l \frac{d^2y}{dx^2} \eta dx = 0. \quad (6)$$

Solving simultaneously Eqs. (3) and (6) makes it possible to determine  $H(x)$  and the vertical translations  $\eta$ , while Eq. (5) makes possible finding horizontal translations  $u$ .

### §3. Solution of Basic Equations

First we transform Eqs. (3) and (6). We represent the horizontal tension in the form of the equality

$$H(x) = H_0 + \Delta H(x), \quad (7)$$

where  $H_0$  is the horizontal tension at the coordinate origin (in our case at the left support). Quantity  $\Delta H(x)$  is found on the basis of Equality (2) in the form

$$\Delta H(x) = - \int_0^x q_x \left| \frac{dy}{dx} \right| dx. \quad (8)$$

Substituting Expression (7) into Equalities (6) and (3) and performing simple transformations, we get

$$\begin{aligned} \frac{H_0 - H_g}{EF} \int_0^l \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2} dx + \frac{H_g}{EF} \int_0^l \frac{\Delta H(x)}{H_g} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2} dx + \int_0^l \frac{d^2y}{dx^2} \eta dx = 0; \\ \frac{d^2\eta}{dx^2} = -\frac{d^2y}{dx^2} - \frac{g + q_y}{H_0} - \frac{H_0}{H_0 + \Delta H(x)} \left[ 1 - \frac{q_x}{g + q_y} \left| \frac{dy}{dx} \right| \frac{dy}{dx} \right]. \end{aligned}$$

We designate

$$\Psi(x) = \frac{\Delta H(x)}{H_g} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2}, \quad (9)$$

$$\Phi(x) = \frac{H_0}{H_0 + \Delta H(x)} \left[ 1 - \frac{q_x}{g + q_y} \left| \frac{dy}{dx} \right| \frac{dy}{dx} \right]. \quad (10)$$

Further, we evaluate the integral

$$\int_0^l \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} dx = l\mu,$$

where

$$\mu = \left[ \frac{5}{8} + 4 \left( \frac{f}{l} \right)^2 \right] \sqrt{1 + 16 \left( \frac{f}{l} \right)^2} + \frac{3l}{32f} \ln \left[ 4 \frac{f}{l} + \sqrt{1 + 16 \left( \frac{f}{l} \right)^2} \right]. \quad (11)$$

Substituting and making use of the fact that for a quadratic parabola  $\frac{d^2y}{dx^2} = \text{const} = \frac{8f}{l^3}$ , we get

$$\frac{H_0 - H_g}{EF} \mu l + \frac{H_g}{EF} \int_0^l \psi(x) dx - \frac{8f}{l^3} \int_0^l \eta dx = 0; \quad (12)$$

$$\frac{d^2\eta}{dx^2} = -\frac{d^2y}{dx^2} - \frac{g + q_y}{H_0} \Phi(x). \quad (13)$$

We now apply numerical integrating using the integral matrix. We break up the span into  $n$  equal parts. According to the equation obtained, we have

$$\vec{\eta} = \vec{\sigma} \eta_0 + \vec{\tau} \eta'_0 + \Omega^2 \vec{y}^*.$$

Taking into account that  $\eta_0 = 0$ , we get

$$\vec{\eta} = \vec{\tau} \eta'_0 + \Omega^2 \vec{y}^*. \quad (14)$$

To determine  $\eta'_0$  we use the condition that when  $x=l$   $\eta=0$ . Writing out the last coordinate of vector  $\vec{\eta}$  and equating it to zero, and also taking into account the fact that the last coordinate of vector  $\vec{\tau}$  is equal to  $n$ , and denoting the last row of matrix  $\Omega^2$  by  $\omega_2$ , we find

$$n\eta'_0 + \omega_2 \vec{y}^* = 0.$$

Consequently:

$$\eta'_0 = -\frac{1}{n} \omega_2 \vec{y}^*.$$

Substituting this expression into Eq. (14), we get

$$\vec{\eta} = -L \vec{y}^*, \quad (15)$$

where

$$L = \frac{1}{n} \vec{\tau} \omega_2 - \Omega^2. \quad (16)$$

We write Eq. (13) in vector form

$$\vec{\eta}^* = -\vec{y}^* - \lambda \frac{g}{H_g} G \vec{\Phi}. \quad (17)$$

In this expression

$$\lambda = \frac{H_g}{H_0} \quad (18)$$

Diagonal matrix  $G$  has the form

$$G = \begin{pmatrix} \left(\frac{g+q_y}{g}\right)_1 & & \\ & \left(\frac{g+q_y}{g}\right)_2 & \\ & \dots & \\ & & \left(\frac{g+q_y}{g}\right)_n \end{pmatrix} \quad (19)$$

while vector  $\vec{\Phi}$  has elements which are obtained from the formula

$$\vec{\Phi} = \frac{H_0}{H_0 + \Delta H(x)} \left[ 1 - \frac{q_x}{g+q_y} |y'| y' \right]. \quad (20)$$

Using Equality (15) and taking into account (17) and also the fact that  $\frac{g}{H_g} = \frac{8f}{l^2}$ , we get

$$\vec{\eta} = -\dot{y} + \lambda \frac{8f}{l^2} LG \vec{\Phi}. \quad (21)$$

Together with this, using the last row of integral matrix  $\Omega$ , which is denoted by  $\omega$ , we get

$$\int_0^l \eta dx = \omega \vec{\eta} = \omega \left( -\dot{y} + \lambda \frac{8f}{l^2} LG \vec{\Phi} \right).$$

Substituting this integral into Eq. (6) and performing all the transformations, we get a quadratic equation for  $\lambda$ :

$$\lambda^2 - 2m_1 \lambda - m_2 = 0.$$

The root of this equation which is of interest is

$$\lambda = m_1 + \sqrt{m_1^2 + m_2},$$

where

$$\begin{aligned} m_1 &= \frac{1}{2\Delta} (1-a); \quad m_2 = \frac{a}{\Delta} \cdot \frac{\mu}{\mu-v}; \\ \Delta &= \frac{8f}{l^2 \omega \vec{y}} \omega LG \vec{\Phi}; \quad a = \frac{H_g l^2 (\mu-v)}{8fEF \omega \vec{y}}; \\ v &= \frac{1}{l} \omega \vec{\Phi}; \quad \vec{y}' = \frac{\Delta H(x)}{H_g} [1 + (y')^2]^{1/2}. \end{aligned}$$

It follows from these formulas that the problem is not solved in the closed form, since in determining  $\lambda$  use is made of values which depend on the sought horizontal tension, which is a part of vector  $\vec{\Phi}$ .

We determine  $\lambda$  by the method of successive approximations. In the first approximation we set all the coordinates of vector  $\vec{\Phi}$

equal to unity. Upon finding  $\lambda$  and then the horizontal tension  $H_0 = \frac{H_g}{\lambda}$ , we calculate a new vector  $\vec{\Phi}$  and then repeat the entire process until the value of  $\lambda$  as well as the coordinates of vector  $\vec{\Phi}$  in two last approximations become identical. Solution of specific problems showed that it is necessary to make from 4 to 6 approximations.

After the horizontal tension is found, the vector of vertical translations is found from Formula (21).

In determining vector  $u$  the integrals contained in Formula (5) will be evaluated by the numerical method. For our boundary conditions we have

$$\vec{u} = \Omega \left( \frac{H(x) - H_g}{EF} \vec{\gamma}_u - \frac{8f}{l^2} \vec{\eta} \right) - \vec{y}' \vec{\eta}, \quad (22)$$

where

$$\vec{\gamma}_u = [1 + (y')^2]^{1/2}.$$

#### §4. Example

We consider a numerical example for the string shown in Fig. 4. Let  $l=160$  m,  $f=20$  m. The horizontal load is uniformly distributed over the entire height and acts on both halves of the string in the form of a wind  $q_x=0.09$  t/m. The vertical load has a triangular distribution and it is assumed that at the right support it is 1.2 t/m, while  $EF=0.2 \cdot 10^6$  t.

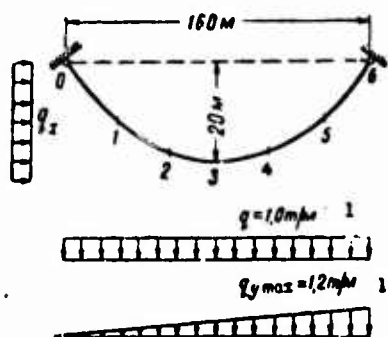


Fig. 4. 1) t/m.

We break up the span into 6 equal parts. The integral matrix and matrix  $L$  for this case are:

$$\Omega = \frac{1}{3120} \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1351 & 2034 & -336 & 90 & -24 & 6 & -1 \\ 1198 & 3732 & 1452 & -180 & 48 & -12 & 2 \\ 1239 & 3486 & 3216 & 1590 & -216 & 54 & -9 \\ 1228 & 3552 & 2952 & 3360 & 1548 & -192 & 32 \\ 1231 & 3534 & 3024 & 3090 & 3336 & 1506 & -121 \\ 1230 & 3540 & 3000 & 3180 & 3000 & 3540 & 1230 \end{vmatrix}$$

$$L = \frac{d^2}{3 \cdot 6240} \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2330 & 12252 & 13548 & 8922 & 6456 & 2910 & 382 \\ 1318 & 12468 & 22746 & 19236 & 12396 & 5988 & 728 \\ 1179 & 8604 & 19638 & 25398 & 19638 & 8604 & 1179 \\ 728 & 5988 & 12396 & 19236 & 22746 & 12468 & 1318 \\ 382 & 2910 & 6456 & 8922 & 13548 & 12252 & 2330 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

This problem was solved on a computer. We present the final result for the example at hand:

$$\lambda = 0,62356$$

$$\vec{\Phi} = \begin{bmatrix} 0,97750 \\ 0,99555 \\ 1,00449 \\ 1,00707 \\ 1,00927 \\ 1,01530 \\ 1,02464 \end{bmatrix}; \quad \vec{H} = \begin{bmatrix} 256,260 \\ 255,260 \\ 254,660 \\ 254,460 \\ 254,260 \\ 253,660 \\ 252,660 \end{bmatrix}; \quad \vec{\eta} = \begin{bmatrix} 0,00000 \\ -0,90634 \\ -0,65874 \\ 0,12693 \\ 0,88553 \\ 1,05138 \\ 0,00000 \end{bmatrix}; \quad \vec{u} = \begin{bmatrix} 0,00000 \\ 0,40529 \\ 0,37143 \\ 0,32137 \\ 0,39399 \\ 0,43417 \\ 0,00009 \end{bmatrix}$$

The last coordinate of vector  $\vec{u}$  should be 0. In our case it is 0.00009. This shows that the calculations are highly exact. When  $EF = \infty$  it was found that

$$\lambda = 0,625000$$

$$\vec{H} = \begin{bmatrix} 256,000 \\ 255,000 \\ 254,400 \\ 254,200 \\ 254,000 \\ 253,400 \\ 252,400 \end{bmatrix}; \quad \vec{\eta} = \begin{bmatrix} 0,00000 \\ -0,96138 \\ -0,75976 \\ -0,00001 \\ 0,75969 \\ 0,96139 \\ 0,00000 \end{bmatrix}; \quad \vec{u} = \begin{bmatrix} 0,00000 \\ 0,41195 \\ 0,37589 \\ 0,31554 \\ 0,37588 \\ 0,41196 \\ 0,00001 \end{bmatrix}$$

It is interesting to note that, if we set  $q_x = 0$ ;  $EF = \infty$ , then the horizontal tension is  $H_0 = 256$  t. With a horizontal load and with  $EF = \infty$   $H_0 = 257.8$  t.

The horizontal tension due to a horizontal load was thus transferred equally to both supports. However, it cannot be claimed that this distribution will prevail in all cases of loading.

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#### Footnotes

- 284 <sup>1</sup>The absolute value of  $dy$  should be taken. This must be stipulated, since when abscissa  $x$  is varied,  $dy$  is first positive and then again becomes negative.
- 286 <sup>2</sup>In the integration we should have added an arbitrary constant, but in our case it is zero, since when  $x = 0$ ,  $u = 0$ .
- 287 <sup>3</sup>Collective work edited by A.F. Smirnov. Raschet sooruzheniy s primeneniym vychislitel'nykh mashin [Design of Structures with the Aid of Computers]. Stroyizdat, 1964 [Formula (V.25)].